

Classical realizability: new tools and applications

Guillaume Geoffroy

Under the supervision of Laurent Regnier
I2M, ED 184, Aix-Marseille Université

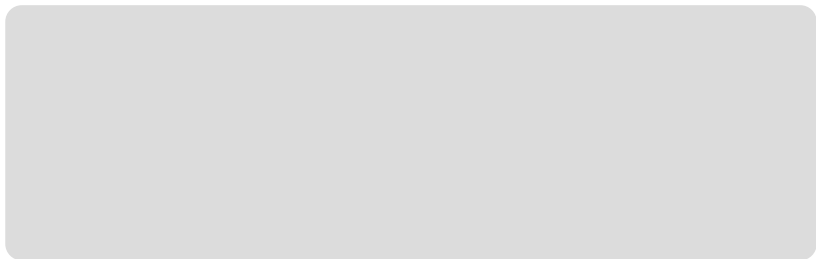
29 March 2019

Realizability & the proof-as-program correspondance

Theorem (Euclid – *Elements*, Book IX – c. 300 BC)

There are infinitely many prime numbers.

Proof.



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Theorem (Euclid – *Elements*, Book IX – c. 300 BC)

There are infinitely many prime numbers.

Proof.

Take any finite list `3,5,7` of prime numbers
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Extract a prime factor q from $A+1$
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Therefore q is not in the list p_1, \dots, p_n



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Realizability & the proof-as-program correspondance

Theorem (Euclid – *Elements*, Book IX – c. 300 BC)

There are infinitely many prime numbers.

Proof.

Take any finite list 3,5,7 of prime numbers

Compute their product $A = 3 \times 5 \times 7 = 105$

Extract a prime factor q from $A + 1$

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Extract a prime factor q from 106

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Take any finite list 3,5,7 of prime numbers

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2 divides 106, therefore 2 does not divide 105

Therefore 2 is not in the list 3,5,7



A program is not (always) a proof

Conjecture (Goldbach, 1742)

For each $n \geq 4$ even, there exist p, q primes such that $n = p + q$.

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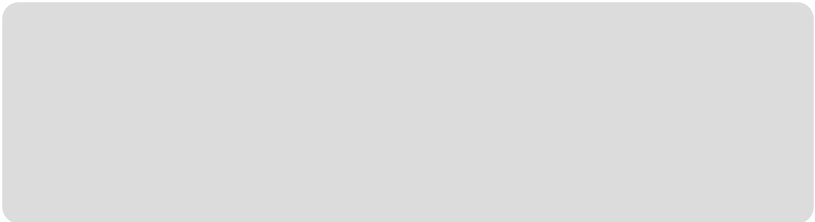
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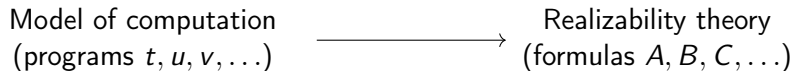
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For all pairs of primes  $(p, q)$  do:  
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  else: keep looking
```

Krivine's classical realizability



Krivine's classical realizability

Model of computation
(programs t, u, v, \dots)

$\xrightarrow{\quad t \Vdash A \quad}$
 $t \text{ realizes } A$

Realizability theory
 $\{ A; \exists t (t \Vdash A) \}$

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Extension of λ -calculus
+ control (call-cc)

\longrightarrow

Extension of ZF

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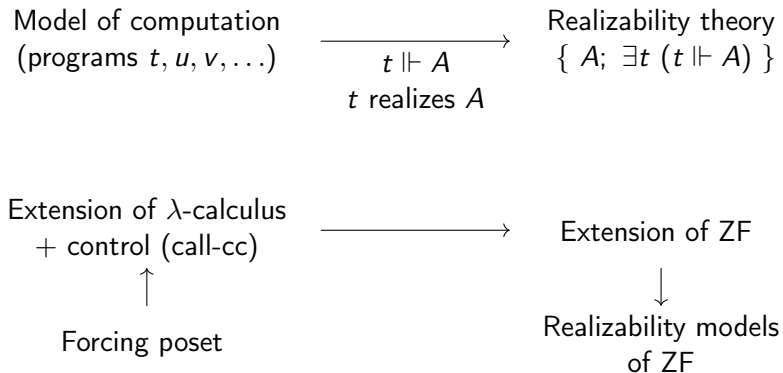
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\downarrow

Realizability models
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Krivine's classical realizability



Krivine's realizability models

Krivine's realizability models: formulas

$$\frac{ZF_\varepsilon}{\varepsilon \subseteq}$$

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Krivine's realizability models: formulas

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$$ZF_\varepsilon \vdash \forall a, b (a \in b \leftrightarrow \exists a' \approx a \ a' \varepsilon b)$$

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Proposition: ZF_ε is a conservative extension of ZF

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$$\frac{\text{Logical}}{\text{Non-logical}} \quad \left| \begin{array}{l} \forall \quad \wedge \quad \vee \quad \rightarrow \quad \dots \\ \cap \quad \cup \quad \leftrightarrow \quad \dots \end{array} \right.$$

Krivine's realizability models: programs & evaluation

$t \star \pi$

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Evaluation rules:

$$\begin{array}{lcl} tu \star \pi & \succ_K & t \star u \bullet \pi & \text{(Push)} \\ \lambda x. t \star u \bullet \pi & \succ_K & t[x := u] \star \pi & \text{(Grab)} \end{array}$$

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Plus transitivity and reflexivity.

Krivine's realizability models: poles

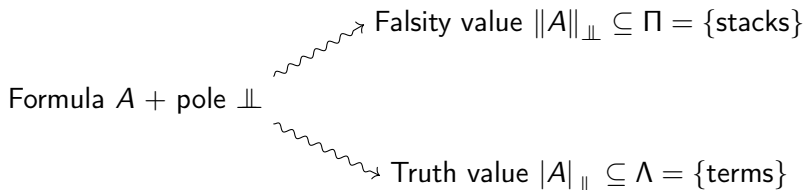
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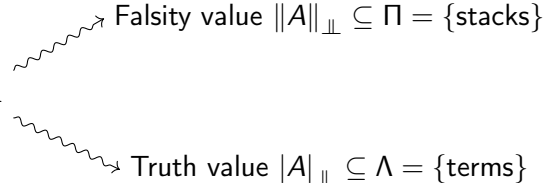
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Formula A + pole \perp 

Falsity value $\|A\|_{\perp} \subseteq \Pi = \{\text{stacks}\}$

Truth value $|A|_{\perp} \subseteq \Lambda = \{\text{terms}\}$

$$|A|_{\perp} = (\|A\|_{\perp})^{\perp} = \{ t; \forall \pi \in \|A\|_{\perp}, t \star \pi \in \perp \}$$

Krivine's realizability models: single-pole models

All poles \longrightarrow Extensions of ZF

$\perp\!\!\!\perp$ \longmapsto Theory($\perp\!\!\!\perp$)

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$\perp\!\!\!\perp$ \longmapsto Theory($\perp\!\!\!\perp$) =
 $\{ A; A \text{ is realized wrt } \perp\!\!\!\perp \}$

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$\perp\!\!\!\perp$ \vdash \longrightarrow $\text{Theory}(\perp\!\!\!\perp) =$
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Proof-like term: a term in which no stack constant (k_π) appears.

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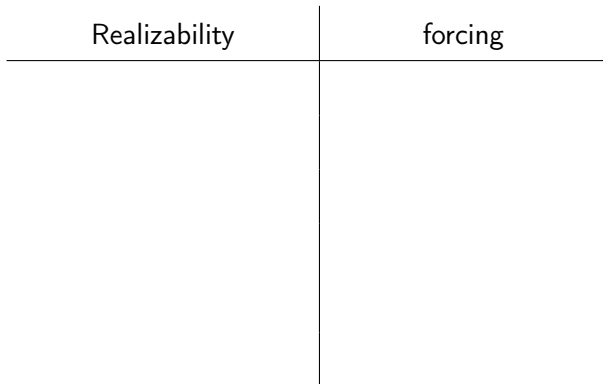
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Adequacy: if $A \vdash_{\text{classical}} B$ and A is realized, then B is realized.

Krivine's realizability models: comparison with forcing



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Realizability	forcing
$\Lambda, \Pi, \Lambda \star \Pi$	P

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$tu, t \cdot \pi, t \star \pi$	$t \wedge u, t \wedge \pi$

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proof-like	\perp
realizes	forces

Krivine's realizability models: $\mathbb{J}2$

First-order formula A
on Boolean Algebras $\xrightarrow{\text{translation}}$ Formula ($\mathbb{J}2 \models A$) of the
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The operation \wedge
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$$\forall x (x = 0) \vee (x = 1) \quad \text{There are only two elements}$$

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$$|\mathbb{J}2 \models A \rightarrow B|_{\perp} = |(\mathbb{J}2 \models A) \rightarrow (\mathbb{J}2 \models B)|_{\perp}$$

$$|\mathbb{J}2 \models \forall x A(x)|_{\perp} = |\mathbb{J}2 \models A(0)|_{\perp} \cap |\mathbb{J}2 \models A(1)|_{\perp}$$

Krivine's realizability models: \mathbb{N}

$$\mathbb{N} \models (0 = 0) \vee (0 = 1)$$

$$\mathbb{N} \models (1 = 0) \vee (1 = 1)$$

$$\mathbb{N} \models \forall x (x = 0) \vee (x = 1)$$

Krivine's realizability models: $\mathbb{I}2$

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Realized by $\lambda x. \lambda y. x$

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Not always realized
(depends on \perp)

Contributions

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Contributions

- ▶ Realizability structures & multi-evaluation relations,
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- ▶ For every λ , $DC_{\hat{\lambda}}$ can be realized,
- ▶ With Laura Fontanella: “ $\hat{\lambda}$ is a cardinal” can be realized.

Multiple poles: realizability structures

Limitation of single-pole models: lack of modularity

- ▶ If \perp grows:

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 - $|a \not\approx b|_{\perp\!\!\!\perp}$ grows
 - $\|a \varepsilon b\|_{\perp\!\!\!\perp}$ grows

Limitation of single-pole models: lack of modularity

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 - $|a \not\in b|_{\perp\!\!\!\perp}$ grows
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 - $|a \in b|_{\perp\!\!\!\perp}$???

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 - $\|a \not\in b\|_{\perp}$ unchanged
 - $|a \not\in b|_{\perp}$ grows
 - $\|a \in b\|_{\perp}$ grows
 - $|a \in b|_{\perp}$???

- ▶ How to combine two poles \perp_1 and \perp_2 ?

Realizability structures

All sets of poles \longrightarrow Extensions of ZF
(realizability structures)

\mathcal{S} \longmapsto Theory(\mathcal{S})

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\mathcal{S} \longmapsto Theory(\mathcal{S}) =
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Realizability structures

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If $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $\text{Theory}(\mathcal{S}_1) \supseteq \text{Theory}(\mathcal{S}_2)$.

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Combining \mathcal{S}_1 and \mathcal{S}_2 : $\mathcal{S}_1 \cap \mathcal{S}_2$.

Multi-evaluation relations

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$$p \succ_K q \Rightarrow p \succ_S q$$

\succ_S reflexive and transitive

Multi-evaluation relations

$p \succ_S q$ iff $\forall \perp \in \mathcal{S}, q \in \perp \Rightarrow p \in \perp$

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 \succ_S reflexive and transitive

\mathcal{S} not uniquely determined by \succ_S

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- ▶ For all P, Q, P', Q' , for all r , *(cut)*
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- ▶ For all P, Q, P', Q' such that $P \succ_{\mathcal{S}} Q$, *(weakening)*
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A *multi-evaluation relation* is a binary relation \succ between sets of processes such that:

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$$\succ \longrightarrow \mathcal{S}_{\succ} = \left\{ \perp; \quad \left. \begin{array}{l} \forall P, Q \text{ s.t. } P \succ Q, \\ Q \subseteq \perp \Rightarrow P \cap \perp \neq \emptyset \end{array} \right\}$$

Taming \mathbb{R}^2 (somewhat)

Adding instructions

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Proof-like term: a term in which no stack constant (k_π) or restricted instruction appears.

Realizing “ \mathbb{N} has fewer than 2^n elements”

Proposition

$\text{Theory}(\mathcal{S}) \vdash \left(\mathbb{N} \models \forall x_1 \dots \forall x_{2^n} \bigvee_{i \neq j} (x_i = x_j) \right)$ *iff*

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$\text{Theory}(\mathcal{S}) \vdash (\mathbb{N} \models \text{Fewer}_{2^n})$ *iff*

Realizing “ $\exists 2$ has fewer than 2^n elements”

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$$\text{Theory}(\mathcal{S}) \vdash \left\{ \begin{array}{l} \cap \quad \color{red}{T} \rightarrow \perp \rightarrow \dots \rightarrow \perp \rightarrow \perp \\ \cap \quad \perp \rightarrow \color{red}{T} \rightarrow \dots \rightarrow \perp \rightarrow \perp \\ \vdots \\ \cap \quad \perp \rightarrow \perp \rightarrow \dots \rightarrow \color{red}{T} \rightarrow \perp \end{array} \right. .$$

$\underbrace{\hspace{10em}}_{n \text{ arguments}}$

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$$\text{Let } \mathcal{S}_{<2^n} = \left\{ \begin{array}{l} \text{for all } u_1, \dots, u_n, \pi, \\ \perp; \text{ if all but one of the } u_i \star \pi \text{ are in } \perp, \\ \text{then } \varphi \star u_1 \bullet \dots \bullet u_n \bullet \pi \text{ is in } \perp \end{array} \right\}.$$

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Let $\mathcal{S}_{<2^n}$ be the largest structure such that for all u_1, \dots, u_n, π ,

$$\begin{array}{ccc} \{\varphi \star u_1 \bullet \dots \bullet u_n \bullet \pi\} & \succ_{\mathcal{S}_{<2^n}} & \{u_i \star \pi; i \neq 1\} \\ \vdots & \vdots & \vdots \\ \{\varphi \star u_1 \bullet \dots \bullet u_n \bullet \pi\} & \succ_{\mathcal{S}_{<2^n}} & \{u_i \star \pi; i \neq n\}. \end{array}$$

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Proposition

$\text{Theory}(\mathcal{S}_{<2^n}) \vdash (\mathbb{N} \models \text{Fewer}_{2^n})$.

Realizing “ \mathbb{N} has at least 2^n elements”

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Let c_{\top} and c_{\perp} be two restricted instructions.

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Let $\mathcal{S}_{\top\perp} = \{ \perp; \text{ for all } \pi, c_{\perp} \star \pi \in \perp \}$.

Realizing “ \mathbb{N} has at least 2^n elements”

Let ξ be any unrestricted instruction other than φ .

Let c_{\top} and c_{\perp} be two restricted instructions.

Let $\mathcal{S}_{\geq 2^n}$ be the largest structure such that for all u, π :

$$\blacktriangleright \{c_{\perp} \star \pi\} \succ_{\mathcal{S}_{\geq 2^n}} \{ \},$$

Realizing “ \mathbb{N} has exactly 2^n elements”

Let $\mathcal{S}_{=2^n} = \mathcal{S}_{<2^{n+1}} \cap \mathcal{S}_{\geq 2^n}$.

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$\text{Theory}(\mathcal{S}_{=2^n}) \not\vdash \perp$.

Realizing any consistent Boolean formula

Theorem

Let A be a Boolean formula. There exists a realizability structure \mathcal{S} such that $\text{Theory}(\mathcal{S}) \vdash \perp \models A$ and $\text{Theory}(\mathcal{S}) \not\vdash \perp$ iff A is satisfiable in Boolean algebras with at least two elements.

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Corollary

Let \mathbb{B} be a Boolean algebra. There exists a realizability structure \mathcal{S} such that $\text{Theory}(\mathcal{S}) \vdash \text{“}\perp \text{ is elementarily equivalent to } \mathbb{B}\text{”}$ and $\text{Theory}(\mathcal{S}) \not\vdash \perp$.

Generalized dependent choices (restricted Zorn's lemma)

A well-ordering instruction

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Let \mathcal{S} be the largest structure such that for all a, b, t, u, v, π :

$$\{\chi \star a \cdot b \cdot t \cdot u \cdot v \cdot \pi\} \succ_{\mathcal{S}} \begin{cases} \{t \star \pi\} & \text{if } a < b \\ \{u \star \pi\} & \text{if } a = b \\ \{v \star \pi\} & \text{if } a > b \end{cases} .$$

Non-extensional choice

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Theory(\mathcal{S}) \vdash “Every family of non-empty sets has a (possibly non \approx -compatible) choice function”.

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b \longmapsto choice(b) ε b

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The generic ordinal $\hat{\lambda}$

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For all $\alpha \leq \lambda$, let $\widehat{\alpha} = \left\{ \left(\widehat{\beta}, \tau_\beta \cdot \pi \right); \beta < \alpha, \pi \text{ stack} \right\}$.

Realizers of $\widehat{\beta} \varepsilon \widehat{\alpha}$: $\sim \begin{cases} \tau_\beta & \text{if } \beta < \alpha \\ \text{none} & \text{if } \beta \geq \alpha \end{cases}$.

Canonical representatives of ordinals

Proposition

$\text{Theory}(\mathcal{S}) \vdash \forall \alpha \in \widehat{\lambda} \forall \beta \in \widehat{\lambda} (\alpha \varepsilon \beta \vee \alpha = \beta \vee \beta \varepsilon \alpha).$

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Theorem

$\text{Theory}(\mathcal{S}) \vdash$ “Every $\hat{\lambda}$ -indexed family of non-empty sets has a \approx -compatible choice function”.

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Theorem

$\text{Theory}(\mathcal{S}) \vdash \text{DC}_{\hat{\lambda}}$ (dependent choice sequences up to length $\hat{\lambda}$.)

Thank you!