

Denotational semantics of probabilistic programs, beyond the discrete case

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Examples of probabilistic programs

Input: random integer r

Output: random integer

$x \leftarrow \text{sample } r$

$y \leftarrow \text{sample uniform } \{0,1\}$

return $x + y$

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Key idea (e.g. *probabilistic coherence spaces*)

- ▶ Linear program \rightsquigarrow linear function

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$$\begin{cases} G(X) & \rightarrow & G(Y) \\ \mu & \mapsto & \sum_{n \in \mathbb{N}} \int_{\vec{x} \in X^n} \varphi_n(\vec{x}) \mu^n(d\vec{x}) \end{cases} \quad \forall n, \varphi_n : X^n \rightarrow G(Y)$$

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- ▶ for any instance X, Y , $\{f : X \rightarrow Y; f \text{ linear}\}$ and $\{f : X \rightarrow Y; f \text{ analytic}\}$ can be equipped with this structure.

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In other words: linear map \Rightarrow morphism of G -algebras
 \Leftrightarrow (for X, Y data types) sub-Markov kernel.

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- ▶ a model of linear logic,
- ▶ in which each construction of linear logic has an intuitive interpretation in “probabilistic” terms.

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▶ $\forall \mu \in G(|X|), \exists! \int_{x \in X} x \mu(dx) \in |X|, \forall \eta \in |X^\perp|,$

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- ▶ $\mu \mapsto \int_{x \in X} x \mu(dx)$ and $(x_n)_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} x_n$ are measurable,
- ▶ and symmetrically (exchanging $|X|$ and $|X^\perp|$).

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- ▶ $\forall f \in |\underline{A}^\perp|, \forall \mu \in |\underline{A}|,$

$$f \mu = \int_{a \in A} f(a) \mu(da)$$

Dual space

Given a linear QBS $\mathcal{X} = (|\mathcal{X}|, |\mathcal{X}^\perp|, (\eta, \mathbf{x}) \mapsto \eta \mathbf{x})$, we define its dual $\mathcal{X}^\perp = (|\mathcal{X}^\perp|, |\mathcal{X}|, (\mathbf{x}, \eta) \mapsto \eta \mathbf{x})$.

Two particular cases

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 - ▶ $|\mathbb{W}| = |\mathbb{W}^\perp| = [0, 1]$
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- ▶ The additive unit $\mathbb{0} \approx \underline{\emptyset}$
 - ▶ $|\mathbb{0}| = |\mathbb{0}^\perp| = \{0\}$

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Linear maps commute with integrals and sums

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- ▶ for all $(x_n)_{n \in \mathbb{N}} \in |X|^{\mathbb{N}}$, if $\sum_{n \in \mathbb{N}} x_n$ is defined, then

$$f \left(\sum_{n \in \mathbb{N}} x_n \right) = \sum_{n \in \mathbb{N}} f(x_n)$$

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▶ $(\mu_n)_{n \in \mathbb{N}} f = \sum_{n \in \mathbb{N}} \int \eta f_X \mu_n(d(x, \eta))$

Multiplicatives

- ▶ $X \multimap \mathbb{W} \simeq X^\perp$
- ▶ $X \wp Y := X^\perp \multimap Y \simeq Y^\perp \multimap X$
- ▶ $X \otimes Y := (X \multimap Y \multimap \mathbb{W}) \multimap \mathbb{W}$
- ▶ $\underline{A} \otimes \underline{B} \simeq \underline{A \times B}$
- ▶ $(\text{QbsLin}, \mathbb{W}, \otimes, \multimap)$ monoidal closed

The cartesian product &

For all linear QBSs $(\mathcal{X}_n)_{n \in \mathbb{N}}$, we define $\&_{n \in \mathbb{N}} \mathcal{X}_n$

$$\blacktriangleright \left| \&_{n \in \mathbb{N}} \mathcal{X}_n \right| = \prod_{n \in \mathbb{N}} |\mathcal{X}_n|$$

$$\blacktriangleright \left| (\&_{n \in \mathbb{N}} \mathcal{X}_n)^\perp \right| = \left\{ (\eta_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} |\mathcal{X}_n^\perp|; \forall (\mathbf{x}_n)_{n \in \mathbb{N}}, \sum_{n \in \mathbb{N}} \eta_n \mathbf{x}_n \leq \mathbf{1} \right\}$$

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The coproduct \oplus

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For all QBSs A, B , $\underline{A} \oplus \underline{B} \simeq \underline{A + B}$

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Proposition

- ▶ $f_!$ is unique
- ▶ $g \circ f = (g_! \circ_{\text{Kleisli}_!} f_!) \circ \nabla$

Arbitrary fixed points

Proposition

- ▶ any analytic function $f : X \rightarrow X$ has a least fixed point $\text{fix } f \in X$,
- ▶ the map $\left\{ \begin{array}{l} \{f : X \rightarrow X \text{ analytic}\} \rightarrow X \\ f \mapsto \text{fix } f \end{array} \right.$ is analytic.