Differential program semantics: sub-modular functions and partial metrics

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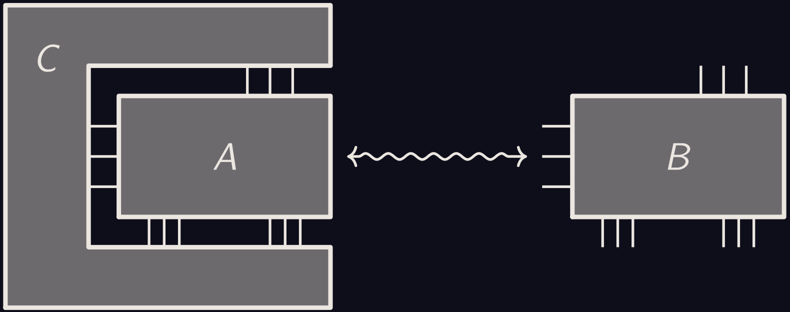
Distance between programs $\partial(t, s)$:

- Compatible with product and function types,
- $\partial(t, t) \not\asymp 0$,
- Asymmetric,
- No triangular inequality.

Differential program semantics



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Approximate denotations

Definition

An *interval space space* $\mathcal{I}$ is the data of:

- a set $|\mathcal{I}|$,
- a subset $\mathcal{I} \subseteq \mathcal{P}(|\mathcal{I}|) \setminus \{\emptyset\}$ closed under non-empty intersections.

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The interval space $[\mathbb{R}]$

- $|[\mathbb{R}]|$,
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Problem: not associative

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- for all $\mathcal{I}$, $\mathcal{J}$, $\mathcal{A}(\mathcal{I}, \mathcal{J})$ is the poset of approximate functions from $\mathcal{I}$ to $\mathcal{J}$.

Exact *vs* aproximate functions

Notation

For all $\varphi \in \mathcal{A}(\mathcal{I}, \mathcal{J})$,

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Lemma For all f, φ:

$$\mathrm{d}f \leq \varphi \Leftrightarrow f \in |\varphi| .$$

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This defines a cartesian product in $\mathcal{A}$:

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Proposition This defines a *lax-expenential*:

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If $f \in |\varphi|$ and $\varphi \to_{\beta\eta} \psi$ then $f \in |\psi|$.

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Diameter spaces

Definition

A *diameter space* is the data of:

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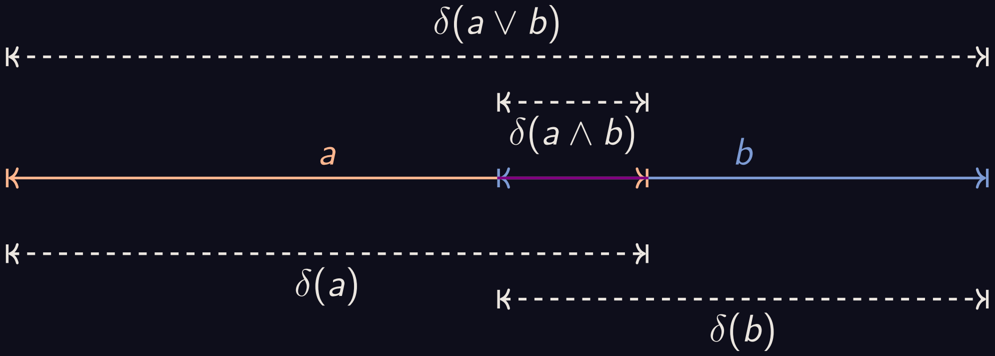

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- $\partial(a, a) \leq \partial(a, b)$,
- $\partial(a, b) = \partial(b, a)$,
- $\partial(a, c) + \partial(b, b) \leq \partial(a, b) + \partial(b, c)$.

Partial metrics on products and exponentials

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- A cartesian lax-closed category whose objects are particular pseudo partial metric spaces.

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