Differential program semantics now with real bi-orthogonality pieces

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DIAPASoN, Unibo

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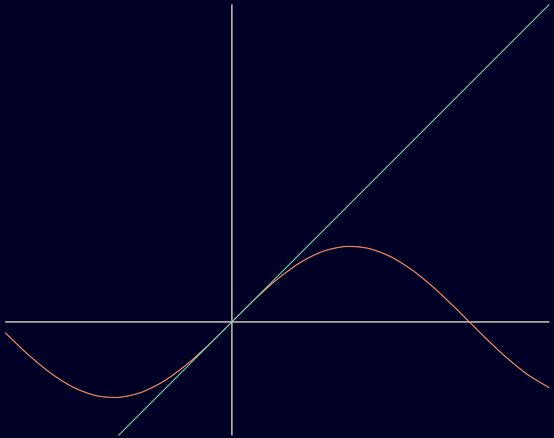
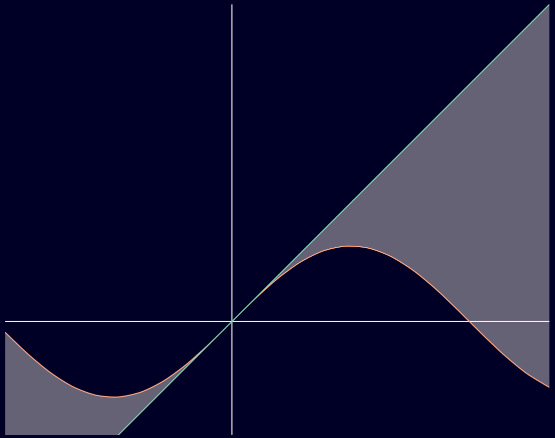
Differential program semantics



Differential program semantics



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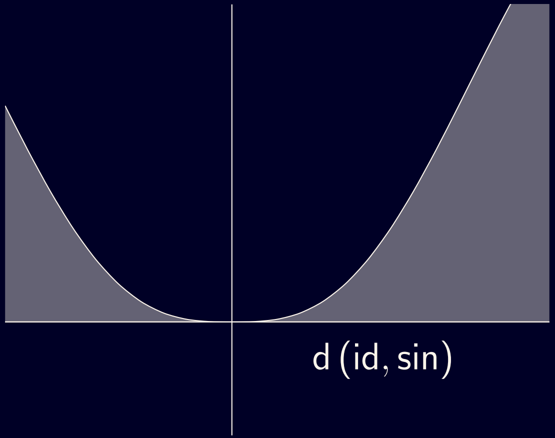
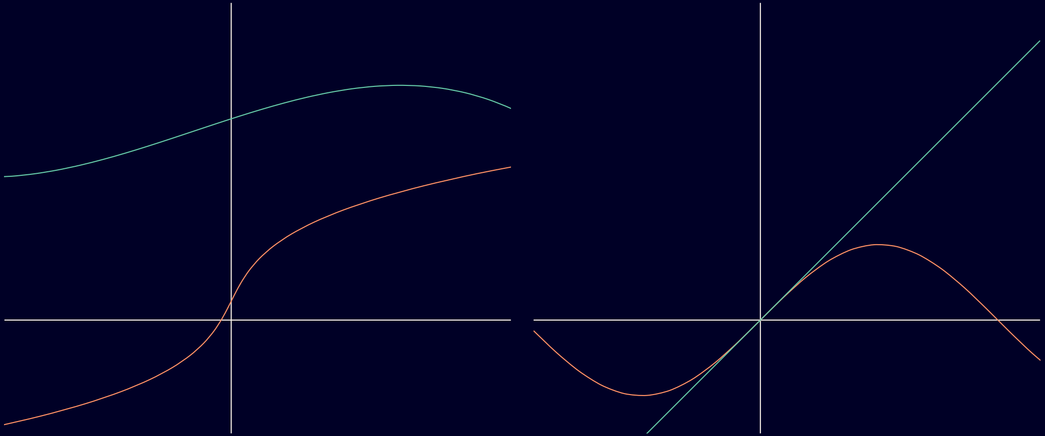


Differential program semantics



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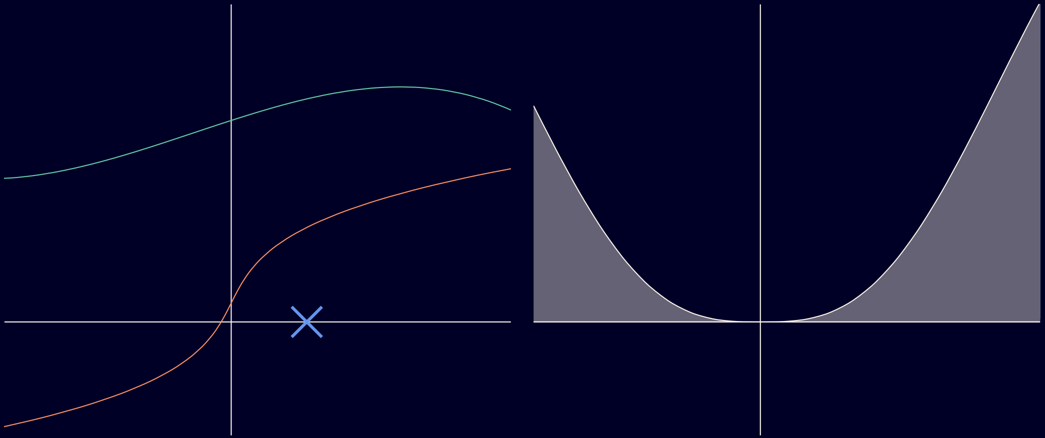

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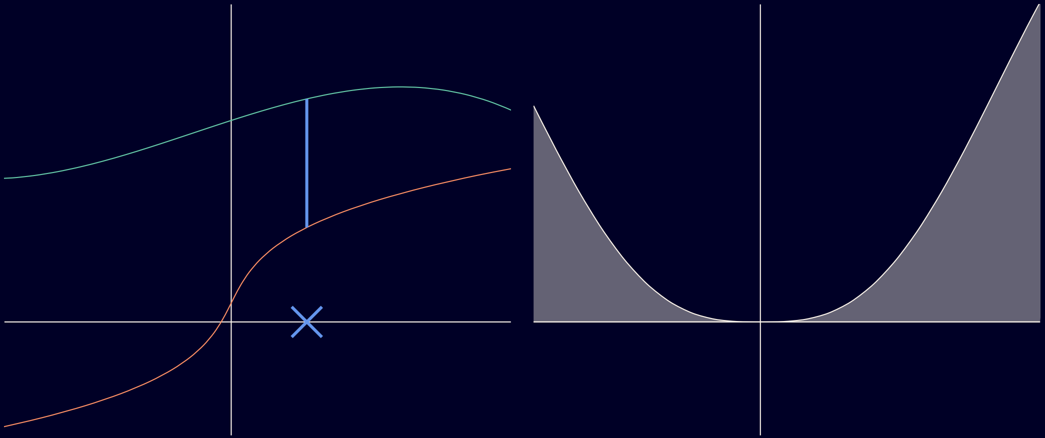

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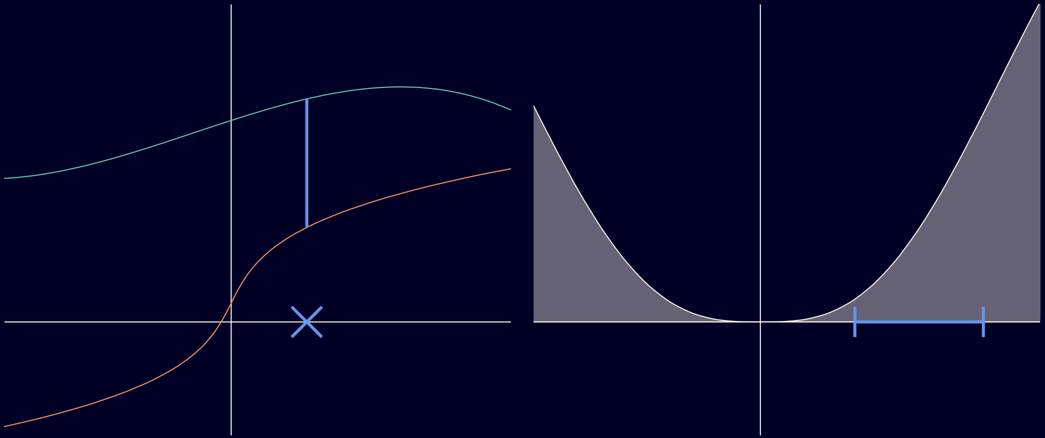

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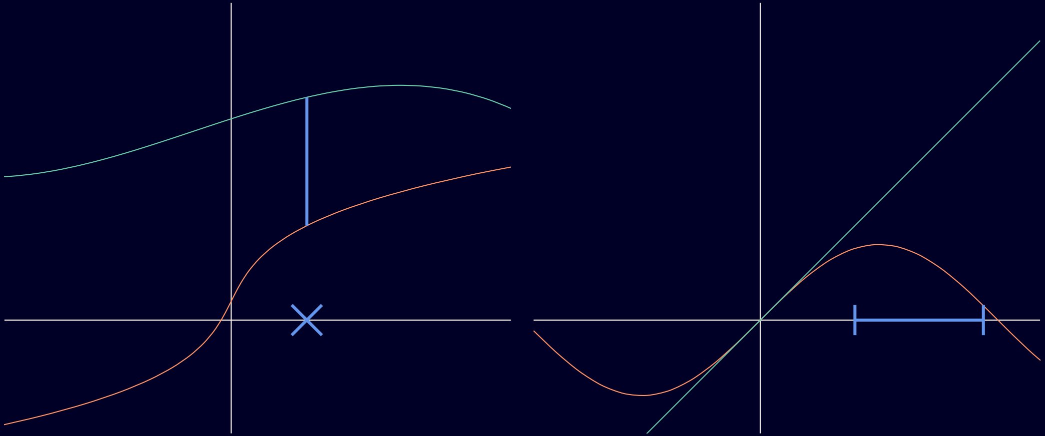

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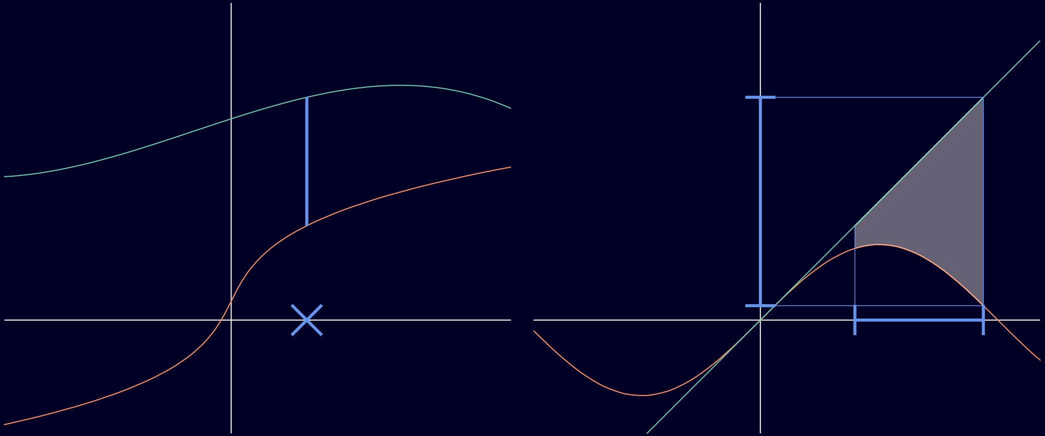

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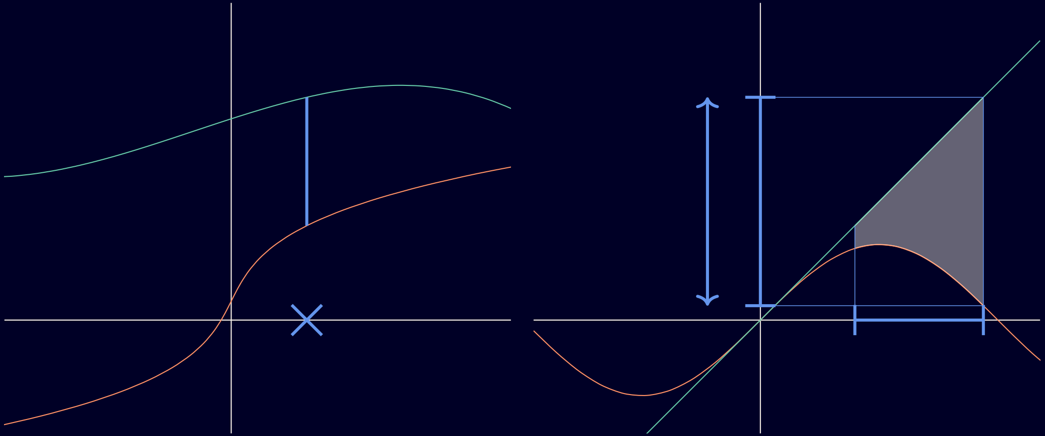

λ-terms – types

$$A, B \quad ::= \quad \mathbb{R}$$

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$$
\begin{array}{rcl}
A, B & ::= & \mathbb{R} \\
& \mid & A \to B
\end{array}
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\begin{aligned}
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Expect: confluence + strong normalization

λ-terms – syntax

$$
\begin{aligned}
t, u \quad ::= \quad & x_A : A \\
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& | & \langle t_A, u_B \rangle : A \times B \\
& | & \rho_L(t_{A \times B}) : A \mid \rho_R(t_{A \times B}) : B
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& | & \rho_L(t_{A \times B}) : A \mid \rho_R(t_{A \times B}) : B \\
& | & f\,(t_{1\mathbb{R}}, \ldots, t_{n\mathbb{R}}) : \mathbb{R} \quad (f : \mathbb{R}^n \to \mathbb{R})
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λ-terms – reduction

- $(\lambda x.\, t)u \to_\beta t\,[x := u]$

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  $(f \circ_j g)\,(t_1, \ldots, t_j, u_1, \ldots u_k, v_1, \ldots, v_l)$

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  $\mathsf{add}\,(1, \mathsf{add}\,(2, 3)) \rightarrow_\beta \mathsf{add}_3\,(1, 2, 3) \rightarrow_\beta^3 6$
- $\rho_L(\langle t, u\rangle) \rightarrow_\beta t$
- $\rho_R(\langle t, u\rangle) \rightarrow_\beta u$

Stacks (*i.e.* tests)

- $\pi_{\mathbb{R}} := I \qquad (I \in \mathcal{I})$

- $t \perp\!\!\!\perp I$ iff $t \to_{\beta}^{*} r \in I$

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Stacks (*i.e.* tests)

- $\pi_{\mathbb{R}} := I \qquad (I \in \mathcal{I})$
- $\pi_{A \to B} = t_A \cdot \pi_B$
- $\pi_{A \times B} = L \cdot \pi_A \mid R \cdot \pi_B$

- $t \perp\!\!\!\perp I$ iff $t \to_\beta^* r \in I$
- $t \perp\!\!\!\perp u \cdot \pi$ iff $tu \perp\!\!\!\perp \pi$
- $t \perp\!\!\!\perp L \cdot \pi$ iff $\rho_L(t) \perp\!\!\!\perp \pi$
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Approximate programs (*i.e.* specifications)

$$[\![A]\!] \; := \; \{X \subseteq \Lambda_A ;\, X^{\perp\!\!\perp\perp\!\!\perp} = X\}$$

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- So $[\![\mathbb{R}]\!] = \{|I|;\ I \text{ closed interval}\}$

Approximate programs – examples

$$[\![A]\!]^* \;:=\; [\![A]\!] \setminus \{\emptyset\}$$

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Example: $[\![\mathbb{R}^n \to \mathbb{R}]\!]$

- For all $F : \mathbb{R} \times \ldots \times \mathbb{R} \to \mathcal{I}^*$, let
  $|F| := \{r_1 \cdot \ldots \cdot r_n \cdot F\,(r_1, \ldots, r_n)\,;\, r_1, \ldots, r_n \in \mathbb{R}\}^{\perp\!\!\!\perp}$

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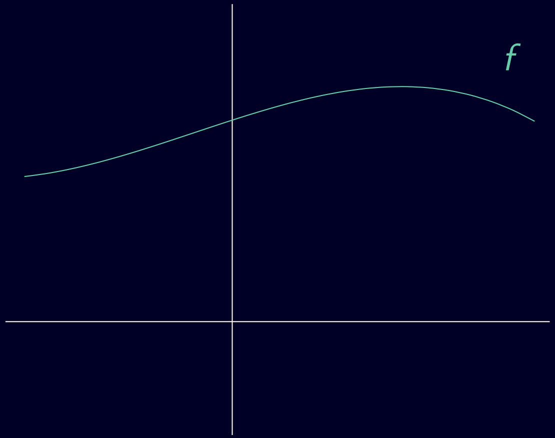
- Then $[\![\mathbb{R} \to \ldots \to \mathbb{R} \to \mathbb{R}]\!]^* = \{|F|\,;\; F \ldots\}$

Approximate programs – examples



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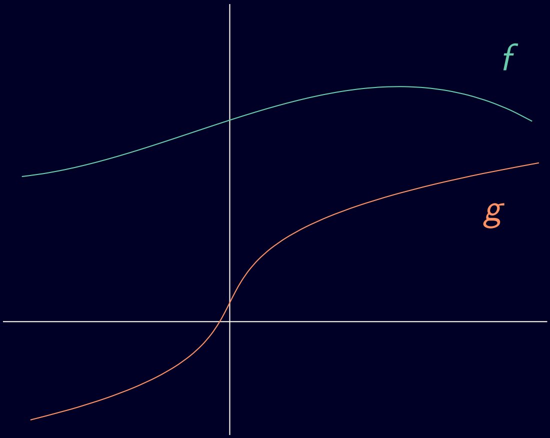

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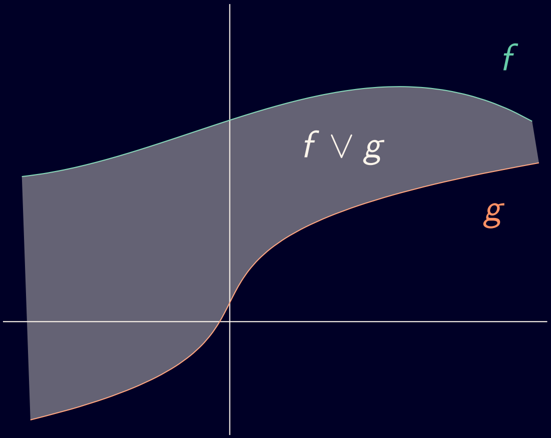

Approximate programs – examples

Example: $[\![A \times B]\!]$

- For all $a \in [\![A]\!]$, $b \in [\![B]\!]$, let
  $|a \times b| := \left\{t;\ t \to_{\beta}^{*} \langle u, v\rangle,\ u \in a, v \in b\right\}$

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- Then $[\![A \times B]\!]^{*} = \{|a \times b|\ ;\ a \in [\![A]\!]^{*}, b \in [\![B]\!]^{*}\}$

Substitution

- $t\,[x_1 : A_1, \ldots, x_n : A_n] : B$

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Substitution

- $t\,[x_1 : A_1, \ldots, x_n : A_n] : B$
- $a_1 \in [\![A_1]\!], \ldots, a_n \in [\![A_n]\!]$
- Then let

$$t\,[x_1 := a_1, \ldots, x_n := a_n] := \left\{ \begin{array}{l} t\,[x_1 := u_1, \ldots, x_n := u_n]\,; \\ u_1 \in a_1, \ldots, u_n \in a_n \end{array} \right\}^{\perp\!\!\!\perp\perp\!\!\!\perp} \in [\![B]\!]$$

Substitution – sub-functoriality

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- $a_1 \in [\![A_1]\!], \ldots, a_m \in [\![A_m]\!]$
- $\rightsquigarrow$ $$\begin{array}{ll} & t\,[u_1, \ldots, u_n]\,[a_1, \ldots, a_m] \\ \subseteq & t\,[u_1\,[a_1, \ldots, a_m], \ldots, u_n\,[a_1, \ldots, a_m]] \end{array}$$

Distances – distance spaces

- $(\!|\mathbb{R}|\!) := \mathbb{R}_+^\infty$
- $(\!|A \times B|\!) := (\!|A|\!) \times (\!|B|\!)$
- $(\!|A \to B|\!) := [\![A]\!] \to (\!|B|\!)$

Distances – diameter function

$\delta_A : [\![A]\!] \to (\!|A|\!)$:

- $\delta_{\mathbb{R}}(|I|) := \text{length}(I)$
- $\delta_{A\times B}(p) := (\delta_A(\rho_L(p)), \delta_B(\rho_R(p)))$
- $\delta_{A\to B}(f)(a) := \delta_B(fa)$

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$d_A(a, b) := \delta_A(a \vee b)$

Distances – sub-modularity

Proposition

If $a \wedge b \neq \emptyset$ then

$$\delta_A(a \vee b) + \delta_A(a \wedge b) \leq \delta_A(a) + \delta_A(b)$$

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Corollary

For all $a, b, c \in [\![A]\!]^*$,

$$\delta_A(a \vee c) + \delta_A(b) \leq \delta_A(a \vee b) + \delta_A(b \vee c)$$

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Corollary

For all $a, b, c \in [\![A]\!]^*$,

$$d_A(a, c) + d_A(b, b) \leq d_A(a, b) + d_A(b, c)$$

Distances – partial metric ?

Proposition?

$[\![A]\!]^*$ is a partial metric space:

- $d_A(a, a) \leq d_A(a, b)$
- $d_A(a, b) = d_A(b, a)$
- $d_A(a, c) + d_A(b, b) \leq d_A(a, b) + d_A(b, c)$
- if $d_A(a, a) = d_A(a, b) = d_A(b, b)$, then $a = b$

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Distances – locally finite distances

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$$[\![A]\!]_f = \{a \in [\![A]\!]\,;\ \delta_A(a) \in (\!|A|\!)_f\}$$

Distances – partial metric

Proposition

$[\![A]\!]^*$ is almost a partial metric space:

- $d_A(a, a) \leq d_A(a, b)$
- $d_A(a, b) = d_A(b, a)$
- $d_A(a, c) + d_A(b, b) \leq d_A(a, b) + d_A(b, c)$
- if $d_A(a, a) = d_A(a, b) = d_A(b, b) \in (\!|A|\!)_f$, then $a = b$