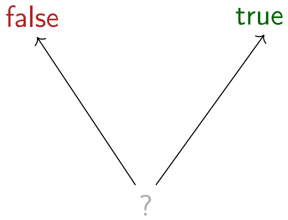
Connecting degrees of parallelism and Boolean algebras through classical realizability

Guillaume Geoffroy

I2M, Aix-Marseille Université

September 20, 2018

Scott's domain of Booleans



Left-first or: $\overrightarrow{\text{or}}$

$y \backslash x$	true	false	?
true	true	true	?
false	true	false	?
?	true	?	?

Left-first or: $\overrightarrow{\text{or}}$

y \ x	true	false	?
true	true	true	?
false	true	false	?
?	true	?	?

$$\lambda x.\lambda y.\ (\text{if}\ \ x)\ y\ \text{true}$$

Right-first or: $\overleftarrow{\text{or}}$

y \ x	true	false	?
true	true	true	true
false	true	false	?
?	?	?	?

$$\lambda x.\lambda y.\ (\text{if}\ \ y)\ x\ \text{true}$$

Parallel or: $\overleftrightarrow{\text{or}}$

y \ x	true	false	?
true	true	true	true
false	true	false	?
?	true	?	?

Parallel or: $\overleftrightarrow{\text{or}}$

y \ x	true	false	?
true	true	true	true
false	true	false	?
?	true	?	?

$$\lambda x.\lambda y.\ ???$$

Voting function: vote

true	true	?	⟼	true
?	true	true	⟼	true
true	?	true	⟼	true
false	false	?	⟼	false
?	false	false	⟼	false
false	?	false	⟼	false

Voting vs parallel or

Voting vs parallel or

$$\overleftrightarrow{\text{or}} = \lambda x.\lambda y.\ \text{vote}\ \left(\overrightarrow{\text{or}}\ x\ y\right)\ \left(\overleftarrow{\text{or}}\ x\ y\right)\ \text{true}$$

Voting vs parallel or

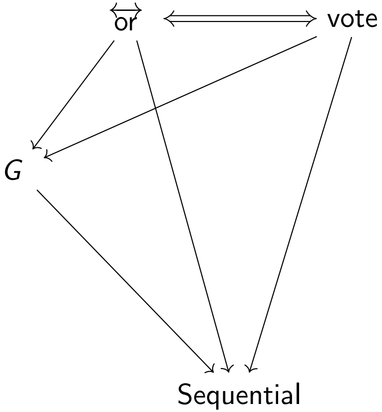
$$\overleftrightarrow{\text{or}} = \lambda x.\lambda y.\ \text{vote}\ \left(\overrightarrow{\text{or}}\ x\ y\right)\ \left(\overleftarrow{\text{or}}\ x\ y\right)\ \text{true}$$

$$\text{vote} = \overleftrightarrow{\text{or}}\left(\overleftrightarrow{\text{or}}\left(\overleftrightarrow{\text{and}}(x,y), \overleftrightarrow{\text{and}}(y,z)\right), \overleftrightarrow{\text{and}}(z,x)\right)$$

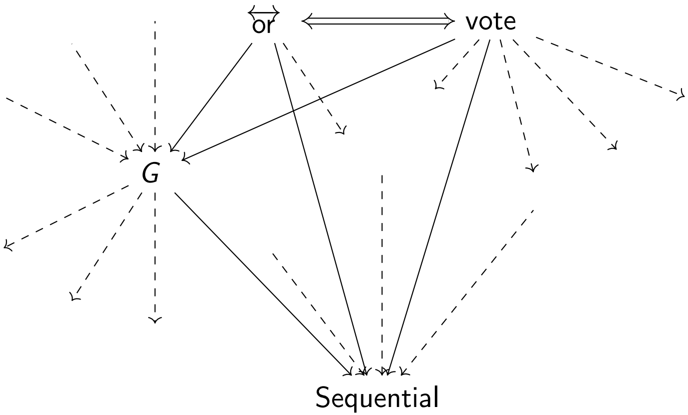
Gustave's function: G

true	false	?	$\longmapsto$	true
?	true	false	$\longmapsto$	true
false	?	true	$\longmapsto$	true
true	true	true	$\longmapsto$	false
false	false	false	$\longmapsto$	false

Degrees of parallelism



Degrees of parallelism



The language of Boolean algebras

Boolean terms: $a, b \quad ::= \quad x \;\mid\; 0 \;\mid\; 1 \;\mid\; a \vee b \;\mid\; a \wedge b \;\mid\; \neg a$

Boolean formulas: $A, B \quad ::= \quad a \neq b \;\mid\; A \to B \;\mid\; \forall x\, A$

The language of Boolean algebras

Boolean terms: $a, b \quad ::= \quad x \mid 0 \mid 1 \mid a \vee b \mid a \wedge b \mid \neg a$

Boolean formulas: $A, B \quad ::= \quad a \neq b \mid A \to B \mid \forall x\ A$

- $(0 \neq 1) \equiv (1 \neq 0) \equiv \top$ (at least two elements)
- $(0 \neq 0) \equiv (1 \neq 1) \equiv \bot$

The language of Boolean algebras

Boolean terms: $a, b \quad ::= \quad x \mid 0 \mid 1 \mid a \vee b \mid a \wedge b \mid \neg a$

Boolean formulas: $A, B \quad ::= \quad a \neq b \mid A \to B \mid \forall x\ A$

- $(0 \neq 1) \equiv (1 \neq 0) \equiv \top$ (at least two elements)
- $(0 \neq 0) \equiv (1 \neq 1) \equiv \bot$
- $a(\overline{x}) \equiv b(\overline{x})$ if $a(\overline{c}) = b(\overline{c})$ for all $\overline{c} \in \{0, 1\}$

The language of Boolean algebras

Associativity of $\wedge$: $\forall x\ \forall y\ \forall z\ \ ((x \wedge y) \wedge z \neq x \wedge (y \wedge z) \to \bot)$

The language of Boolean algebras

Associativity of $\wedge$: $\forall x\ \forall y\ \forall z\ \ ((x \wedge y) \wedge z \neq x \wedge (y \wedge z) \to \bot)$

At most 2 elements: $\forall x\ (x \neq 0 \to x \neq 1 \to \bot)$

The language of Boolean algebras

Associativity of $\wedge$: $\forall x\ \forall y\ \forall z\ \ ((x \wedge y) \wedge z \neq x \wedge (y \wedge z) \to \bot)$

At most 2 elements: $\forall x\ (x \neq 0 \to x \neq 1 \to \bot)$

At most 4 elements:
$\forall x\, \forall y\, \forall z\, (x \neq 0 \to y \neq 0 \to z \neq 0 \to ((x{\wedge}y){\vee}(y{\wedge}z){\vee}(z{\wedge}x)) \neq 0)$

The language of Boolean algebras

Associativity of $\wedge$: $\forall x\ \forall y\ \forall z\ \ ((x \wedge y) \wedge z \neq x \wedge (y \wedge z) \to \bot)$

At most 2 elements: $\forall x\ (x \neq 0 \to x \neq 1 \to \bot)$

At most 4 elements:
$\forall x\,\forall y\,\forall z\,(x \neq 0 \to y \neq 0 \to z \neq 0 \to ((x{\wedge}y){\vee}(y{\wedge}z){\vee}(z{\wedge}x)) \neq 0)$

At least 4 elements: $\forall x\ (x \neq 0 \to x \neq 1 \to \bot) \to \bot$

Formulas with intersections

$$
\begin{array}{rcl}
A, B & ::= & \top \mid \bot \\
& \mid & A \to B \\
& \mid & A \cap B
\end{array}
$$

Formulas with intersections

$$A, B \quad ::= \quad \top \mid \bot \\ \mid \; A \to B \\ \mid \; A \cap B$$

Identifications:

- $A \cap B \equiv B \cap A$, $A \cap (B \cap C) \equiv (A \cap B) \cap C$, etc.

Formulas with intersections

$$
\begin{array}{rcl}
A, B & ::= & \top \mid \bot \\
& \mid & A \to B \\
& \mid & A \cap B
\end{array}
$$

Identifications:

- $A \cap B \equiv B \cap A$, $A \cap (B \cap C) \equiv (A \cap B) \cap C$, etc.
- $(A \to B) \cap (A \to C) \equiv A \to (B \cap C)$

Formulas with intersections

$$\begin{array}{rcl} A, B & ::= & \top \mid \bot \\ & \mid & A \to B \\ & \mid & A \cap B \end{array}$$

Identifications:

- $A \cap B \equiv B \cap A$, $A \cap (B \cap C) \equiv (A \cap B) \cap C$, etc.
- $(A \to B) \cap (A \to C) \equiv A \to (B \cap C)$
- $\top \to \bot \equiv \bot$

Formulas with intersections

$$
\begin{array}{rcl}
A, B & ::= & \top \mid \bot \\
 & \mid & A \to B \\
 & \mid & A \cap B
\end{array}
$$

Identifications:

- $A \cap B \equiv B \cap A$, $A \cap (B \cap C) \equiv (A \cap B) \cap C$, etc.
- $(A \to B) \cap (A \to C) \equiv A \to (B \cap C)$
- $\top \to \bot \equiv \bot$
- $A \to \top \equiv \top$

Formulas with intersections

$$
\begin{array}{rcl}
A, B & ::= & \top \mid \bot \\
& \mid & A \to B \\
& \mid & A \cap B
\end{array}
$$

Identifications:

- $A \cap B \equiv B \cap A$, $A \cap (B \cap C) \equiv (A \cap B) \cap C$, etc.
- $(A \to B) \cap (A \to C) \equiv A \to (B \cap C)$
- $\top \to \bot \equiv \bot$
- $A \to \top \equiv \top$

Subtyping:

- $\bot \leqq A$, $A \leqq \top$
- $A \cap B$ greatest lower bound
- if $A \geqq A'$ and $B \leqq B'$, then $(A \to B) \leqq (A' \to B')$

To the simply typed λ_c-calculus...

$$
\begin{array}{rcl}
A, B & ::= & \top \;\mid\; \bot \\
& \mid & A \to B \\
& \mid & A \cap B
\end{array}
$$

To the simply typed λ_c-calculus...

$$
\begin{array}{rcl}
A, B & ::= & \top \mid \bot \\
& \mid & A \to B \\
& \mid & A \cap B
\end{array}
$$

$$[\![\lambda x^o.\ x]\!] \equiv \bot \to \bot$$

To the simply typed λ_c-calculus...

$$
\begin{array}{rcl}
A, B & ::= & \top \mid \bot \\
& \mid & A \to B \\
& \mid & A \cap B
\end{array}
$$

$$[\![\lambda x^o.\ x]\!] \equiv \bot \to \bot$$
$$[\![\lambda x^o.\lambda y^o.\ x]\!] \equiv \bot \to \top \to \bot$$

To the simply typed λ_c-calculus...

$$
\begin{array}{rcl}
A, B & ::= & \top \mid \bot \\
& \mid & A \to B \\
& \mid & A \cap B
\end{array}
$$

$$
\begin{array}{l}
[\![\lambda x^o.\ x]\!] \equiv \bot \to \bot \\
[\![\lambda x^o.\lambda y^o.\ x]\!] \equiv \bot \to \top \to \bot \\
[\![\lambda x^o.\lambda y^o.\ y]\!] \equiv \top \to \bot \to \bot
\end{array}
$$

To the simply typed λ_c-calculus...

$$\begin{array}{rcl} A, B & ::= & \top \mid \bot \\ & \mid & A \to B \\ & \mid & A \cap B \end{array}$$

$$[\![\lambda x^o.\ x]\!] \equiv \bot \to \bot$$
$$[\![\lambda x^o.\lambda y^o.\ x]\!] \equiv \bot \to \top \to \bot$$
$$[\![\lambda x^o.\lambda y^o.\ y]\!] \equiv \top \to \bot \to \bot$$

$$[\![\lambda a^{o\to o\to o}.\lambda b^{o\to o\to o}.\ \lambda x^o.\lambda y^o.\ a\ (b\ x\ y)\ y]\!]$$

To the simply typed λ_c-calculus...

$$A, B \quad ::= \quad \top \mid \bot \mid A \to B \mid A \cap B$$

$$[\![\lambda x^o.\ x]\!] \equiv \bot \to \bot$$
$$[\![\lambda x^o.\lambda y^o.\ x]\!] \equiv \bot \to \top \to \bot$$
$$[\![\lambda x^o.\lambda y^o.\ y]\!] \equiv \top \to \bot \to \bot$$

$$\begin{array}{l}
[\![\lambda a^{o\to o\to o}.\lambda b^{o\to o\to o}.\ \lambda x^o.\lambda y^o.\ a\ (b\ x\ y)\ y]\!] \\
\begin{array}{llcccc}
\equiv & & (\top \to \bot \to \bot) & \to & \top & \to & (\top \to \bot \to \bot) \\
& \cap & (\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot) \\
& \cap & (\bot \to \top \to \bot) & \to & (\top \to \bot \to \bot) & \to & (\top \to \bot \to \bot)
\end{array}
\end{array}$$

To the simply typed λ_c-calculus...

$$
\begin{array}{rcl}
A, B & ::= & \top \mid \bot \\
 & \mid & A \to B \\
 & \mid & A \cap B
\end{array}
$$

$$
\begin{array}{l}
[\![\lambda x^o.\ x]\!] \equiv \bot \to \bot \\
[\![\lambda x^o.\lambda y^o.\ x]\!] \equiv \bot \to \top \to \bot \\
[\![\lambda x^o.\lambda y^o.\ y]\!] \equiv \top \to \bot \to \bot
\end{array}
$$

$$
\begin{array}{lcccccccc}
\multicolumn{9}{l}{[\![\lambda a^{o\to o\to o}.\lambda b^{o\to o\to o}.\ \lambda x^o.\lambda y^o.\ a\ (b\ x\ y)\ y]\!]} \\
\equiv & & (\top \to \bot \to \bot) & \to & \top & \to & (\top \to \bot \to \bot) \\
 & \cap & (\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot) \\
 & \cap & (\bot \to \top \to \bot) & \to & (\top \to \bot \to \bot) & \to & (\top \to \bot \to \bot) \\
\equiv & \overrightarrow{\text{or}}
\end{array}
$$

… and beyond !

$$
\begin{array}{rcccccc}
\overrightarrow{\text{or}} & \equiv & & (\top \to \bot \to \bot) & \to & \top & \to & (\top \to \bot \to \bot) \\
& & \cap & (\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot) \\
& & \cap & (\bot \to \top \to \bot) & \to & (\top \to \bot \to \bot) & \to & (\top \to \bot \to \bot) \\
& \equiv & \multicolumn{5}{l}{[\![\lambda a^{o\to o\to o}.\lambda b^{o\to o\to o}.\ \lambda x^{o}.\lambda y^{o}.\ a\ (b\ x\ y)\ y]\!]}
\end{array}
$$

... and beyond !

$$\begin{array}{rlcccccc}
\overrightarrow{\text{or}} & \equiv & & (\top \to \bot \to \bot) & \to & \top & \to & (\top \to \bot \to \bot) \\
& & \cap & (\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot) \\
& & \cap & (\bot \to \top \to \bot) & \to & (\top \to \bot \to \bot) & \to & (\top \to \bot \to \bot) \\
& \equiv & \multicolumn{6}{l}{[\![\lambda a^{o\to o\to o}.\lambda b^{o\to o\to o}.\ \lambda x^{o}.\lambda y^{o}.\ a\ (b\ x\ y)\ y]\!]}
\end{array}$$

$$\begin{array}{rlcccccc}
\overleftrightarrow{\text{or}} & \equiv & & (\top \to \bot \to \bot) & \to & \top & \to & (\top \to \bot \to \bot) \\
& & \cap & (\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot) \\
& & \cap & \top & \to & (\top \to \bot \to \bot) & \to & (\top \to \bot \to \bot)
\end{array}$$

… and beyond !

$$
\begin{aligned}
\overrightarrow{\text{or}} \;\equiv\;& \quad (\top \to \bot \to \bot) \to \top \to (\top \to \bot \to \bot) \\
& \cap\ (\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \\
& \cap\ (\bot \to \top \to \bot) \to (\top \to \bot \to \bot) \to (\top \to \bot \to \bot) \\
\equiv\;& [\![\lambda a^{o \to o \to o}.\lambda b^{o \to o \to o}.\ \lambda x^{o}.\lambda y^{o}.\ a\ (b\ x\ y)\ y]\!]
\end{aligned}
$$

$$
\begin{aligned}
\overleftrightarrow{\text{or}} \;\equiv\;& \quad (\top \to \bot \to \bot) \to \top \to (\top \to \bot \to \bot) \\
& \cap\ (\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \\
& \cap\ \top \to (\top \to \bot \to \bot) \to (\top \to \bot \to \bot) \\
\not\equiv\;& [\![\ldots]\!]
\end{aligned}
$$

... and beyond !

$$
\begin{aligned}
\overrightarrow{\text{or}} \;\equiv\; & \quad (\top \to \bot \to \bot) \to \top \to (\top \to \bot \to \bot) \\
& \cap\; (\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \\
& \cap\; (\bot \to \top \to \bot) \to (\top \to \bot \to \bot) \to (\top \to \bot \to \bot) \\
\equiv\; & [\![\lambda a^{o \to o \to o}.\lambda b^{o \to o \to o}.\ \lambda x^{o}.\lambda y^{o}.\ a\ (b\ x\ y)\ y]\!]
\end{aligned}
$$

$$
\begin{aligned}
\overleftrightarrow{\text{or}} \;\equiv\; & \quad (\top \to \bot \to \bot) \to \top \to (\top \to \bot \to \bot) \\
& \cap\; (\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \\
& \cap\; \top \to (\top \to \bot \to \bot) \to (\top \to \bot \to \bot) \\
\not\equiv\; & [\![\ldots]\!]
\end{aligned}
$$

Sequential vs non-sequential

Tweedledum and Tweedledee

$$
\begin{aligned}
A, B \quad ::= \quad & a \neq b \\
\mid \quad & A \to B \\
\mid \quad & \forall x\, A
\end{aligned}
\qquad\qquad
\begin{aligned}
A, B \quad ::= \quad & \top \mid \bot \\
\mid \quad & A \to B \\
\mid \quad & A \cap B
\end{aligned}
$$

Tweedledum and Tweedledee

$$
\begin{array}{rcl}
A, B & ::= & a \neq b \\
& | & A \to B \\
& | & \forall x\ A
\end{array}
\qquad
\begin{array}{rcl}
A, B & ::= & \top \mid \bot \\
& | & A \to B \\
& | & A \cap B
\end{array}
$$

- $(0 \neq 1) \equiv (1 \neq 0) \equiv \top$
- $(0 \neq 0) \equiv (1 \neq 1) \equiv \bot$

Tweedledum and Tweedledee

$$A, B \quad ::= \quad a \neq b \mid A \to B \mid \forall x\, A$$

$$A, B \quad ::= \quad \top \mid \bot \mid A \to B \mid A \cap B$$

- $(0 \neq 1) \equiv (1 \neq 0) \equiv \top$
- $(0 \neq 0) \equiv (1 \neq 1) \equiv \bot$
- $\forall x\, A(x) \equiv A(0) \cap A(1)$

Tweedledum and Tweedledee

$$A, B \quad ::= \quad a \neq b \mid A \to B \mid \forall x\, A$$

$$A, B \quad ::= \quad \top \mid \bot \mid A \to B \mid A \cap B$$

- $(0 \neq 1) \equiv (1 \neq 0) \equiv \top$
- $(0 \neq 0) \equiv (1 \neq 1) \equiv \bot$
- $\forall x\, A(x) \equiv A(0) \cap A(1)$

- Logical interpretation (Boolean algebras)
- Computational interpretation (simply typed λ_c-calculus, degrees of parallelism)

Translating $\overrightarrow{\text{or}}$

$$
\begin{aligned}
\overrightarrow{\text{or}} \equiv \; & ((\top \to \bot \to \bot) \to \top \to (\top \to \bot \to \bot)) \\
\cap \; & ((\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \to (\bot \to \top \to \bot)) \\
\cap \; & ((\bot \to \top \to \bot) \to (\top \to \bot \to \bot) \to (\top \to \bot \to \bot))
\end{aligned}
$$

Translating $\overrightarrow{\text{or}}$

$$\begin{array}{rlcl} \overrightarrow{\text{or}} \equiv & ((\top \to \bot \to \bot) \to & \top & \to (\top \to \bot \to \bot)) \\ \cap & ((\bot \to \top \to \bot) \to & (\bot \to \top \to \bot) & \to (\bot \to \top \to \bot)) \\ \cap & ((\bot \to \top \to \bot) \to & (\top \to \bot \to \bot) & \to (\top \to \bot \to \bot)) \end{array}$$

$$\forall x\ \forall y\ \left(\begin{array}{lc} & (x \neq 0 \to x \neq 1 \to \bot) \\ \to & (y \neq 0 \to y \neq 1 \to \bot) \\ \to & ((x \vee y) \neq 0 \to (x \vee y) \neq 1 \to \bot) \end{array} \right)$$

Translating $\overrightarrow{\text{or}}$

$$\begin{array}{rcccc} \overrightarrow{\text{or}} \equiv & ((\top \to \bot \to \bot) & \to & \top & \to (\top \to \bot \to \bot)) \\ \cap & ((\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot) & \to (\bot \to \top \to \bot)) \\ \cap & ((\bot \to \top \to \bot) & \to & (\top \to \bot \to \bot) & \to (\top \to \bot \to \bot)) \end{array}$$

$$\equiv \forall x\ \forall y\ \left(\begin{array}{rc} & (x \neq 0 \to x \neq 1 \to \bot) \\ \to & (y \neq 0 \to y \neq 1 \to x \neq 0) \\ \to & ((x \vee y) \neq 0 \to (x \vee y) \neq 1 \to \bot) \end{array} \right)$$

Translating $\overrightarrow{\text{or}}$

$$\begin{array}{rlll} \overrightarrow{\text{or}} \equiv & ((\top \to \bot \to \bot) \to & \top & \to (\top \to \bot \to \bot)) \\ \cap & ((\bot \to \top \to \bot) \to & (\bot \to \top \to \bot) & \to (\bot \to \top \to \bot)) \\ \cap & ((\bot \to \top \to \bot) \to & (\top \to \bot \to \bot) & \to (\top \to \bot \to \bot)) \end{array}$$

$$\equiv \forall x \, \forall y \, \begin{pmatrix} & (x \neq 0 \to x \neq 1 \to \bot) \\ \to & (y \neq 0 \to y \neq 1 \to x \neq 0) \\ \to & ((x \vee y) \neq 0 \to (x \vee y) \neq 1 \to \bot) \end{pmatrix}$$

$\rightsquigarrow$ true in every Boolean algebra

Translating $\overleftrightarrow{\text{or}}$

$$\begin{array}{rl} \overleftrightarrow{\text{or}} \equiv & ((\top \to \bot \to \bot) \to \top \to (\top \to \bot \to \bot)) \\ \cap & ((\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \to (\bot \to \top \to \bot)) \\ \cap & (\top \to (\top \to \bot \to \bot) \to (\top \to \bot \to \bot)) \end{array}$$

Translating $\overleftrightarrow{\text{or}}$

$$\begin{array}{rcccccc}
\overleftrightarrow{\text{or}} \equiv & ((\top \to \bot \to \bot) & \to & \top & \to & (\top \to \bot \to \bot)) \\
\cap & ((\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot) & \to & (\bot \to \top \to \bot)) \\
\cap & (\qquad \top & \to & (\top \to \bot \to \bot) & \to & (\top \to \bot \to \bot))
\end{array}$$

$$\forall x\ \forall y\ \left(\begin{array}{rc}
 & (x \neq 0 \to x \neq 1 \to \bot) \\
\to & (y \neq 0 \to y \neq 1 \to x \neq 0) \\
\to & ((x \vee y) \neq 0 \to (x \vee y) \neq 1 \to \bot)
\end{array}\right)$$

Translating $\overleftrightarrow{\text{or}}$

$$
\begin{aligned}
\overleftrightarrow{\text{or}} \equiv\ & ((\top \to \bot \to \bot) \to \top \to (\top \to \bot \to \bot)) \\
\cap\ & ((\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \to (\bot \to \top \to \bot)) \\
\cap\ & (\top \to (\top \to \bot \to \bot) \to (\top \to \bot \to \bot))
\end{aligned}
$$

$$
\equiv \forall x\ \forall y \left(\begin{array}{cc} & (x \neq 0 \to x \neq 1 \to (y \wedge \neg x) \neq 0) \\ \to & (y \neq 0 \to y \neq 1 \to x \neq 0) \\ \to & ((x \vee y) \neq 0 \to (x \vee y) \neq 1 \to \bot) \end{array} \right)
$$

Translating $\overleftrightarrow{\text{or}}$

$$\begin{aligned}\overleftrightarrow{\text{or}} \equiv\ & ((\top \to \bot \to \bot) \to \top \to (\top \to \bot \to \bot)) \\ \cap\ & ((\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \to (\bot \to \top \to \bot)) \\ \cap\ & (\top \to (\top \to \bot \to \bot) \to (\top \to \bot \to \bot))\end{aligned}$$

$$\equiv \forall x\ \forall y\ \begin{pmatrix} & (x \neq 0 \to x \neq 1 \to (y \wedge \neg x) \neq 0) \\ \to & (y \neq 0 \to y \neq 1 \to (x \wedge \neg y) \neq 0) \\ \to & ((x \vee y) \neq 0 \to (x \vee y) \neq 1 \to \bot) \end{pmatrix}$$

Translating $\overleftrightarrow{\text{or}}$

$$
\begin{aligned}
\overleftrightarrow{\text{or}} \equiv \quad & ((\top \to \bot \to \bot) \to \top \to (\top \to \bot \to \bot)) \\
\cap \; & ((\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \to (\bot \to \top \to \bot)) \\
\cap \; & (\top \to (\top \to \bot \to \bot) \to (\top \to \bot \to \bot))
\end{aligned}
$$

$$
\equiv \forall x \; \forall y \; \begin{pmatrix} & (x \neq 0 \to x \neq 1 \to (y \wedge \neg x) \neq 0) \\ \to & (y \neq 0 \to y \neq 1 \to (x \wedge \neg y) \neq 0) \\ \to & ((x \vee y) \neq 0 \to (x \vee y) \neq 1 \to \bot) \end{pmatrix}
$$

$\Leftrightarrow$ there are at most 4 elements

Translating $\overleftrightarrow{\text{or}}$

$$\begin{array}{rl} \overleftrightarrow{\text{or}} \equiv & ((\top \to \bot \to \bot) \to \top \to (\top \to \bot \to \bot)) \\ \cap & ((\bot \to \top \to \bot) \to (\bot \to \top \to \bot) \to (\bot \to \top \to \bot)) \\ \cap & (\top \to (\top \to \bot \to \bot) \to (\top \to \bot \to \bot)) \end{array}$$

$$\equiv \forall x\ \forall y\ \begin{pmatrix} & (x \neq 0 \to x \neq 1 \to (y \wedge \neg x) \neq 0) \\ \to & (y \neq 0 \to y \neq 1 \to (x \wedge \neg y) \neq 0) \\ \to & ((x \vee y) \neq 0 \to (x \vee y) \neq 1 \to \bot) \end{pmatrix}$$

$$\Leftrightarrow \forall x\ \forall y\ \forall z\ \begin{pmatrix} & x \neq 0 \to y \neq 0 \to z \neq 0 \\ \to & ((x \wedge y) \vee (y \wedge z) \vee (z \wedge x)) \neq 0 \end{pmatrix}$$

Realizability to the rescue!

Theorem (Adequacy – $\gimel 2$ is a Boolean algebra)

If A is true in every Boolean algebra, A is sequential.

Realizability to the rescue!

Theorem (Adequacy – $\gimel 2$ is a Boolean algebra)

If A is true in every Boolean algebra, A is sequential.

Theorem (Play-Doh – $\gimel 2$ can be any Boolean algebra)

If A is sequential, A is true in every Boolean algebra.

Realizability to the rescue!

Theorem (Adequacy – $\gimel 2$ is a Boolean algebra)

If A is true in every Boolean algebra, A is sequential.

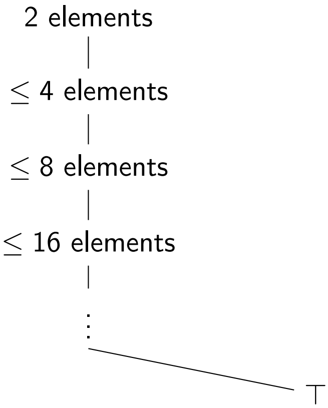
Theorem (Play-Doh – $\gimel 2$ can be any Boolean algebra)

If A is sequential, A is true in every Boolean algebra.

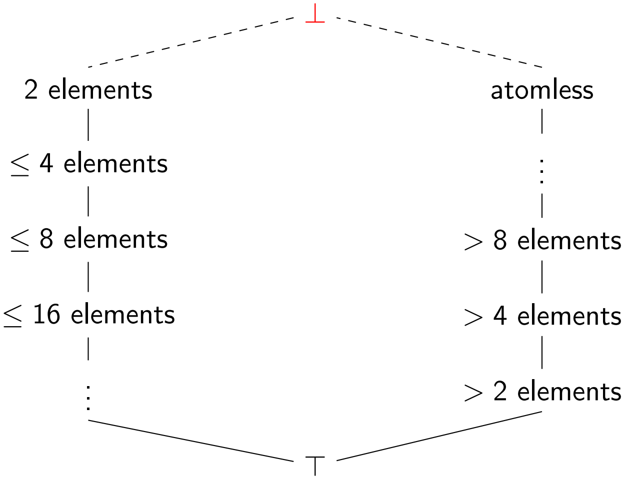
Corollary

A can be simulated from $B_1, \ldots, B_n$ iff $B_1 \to \ldots \to B_n \to A$ is true in every Boolean algebra.

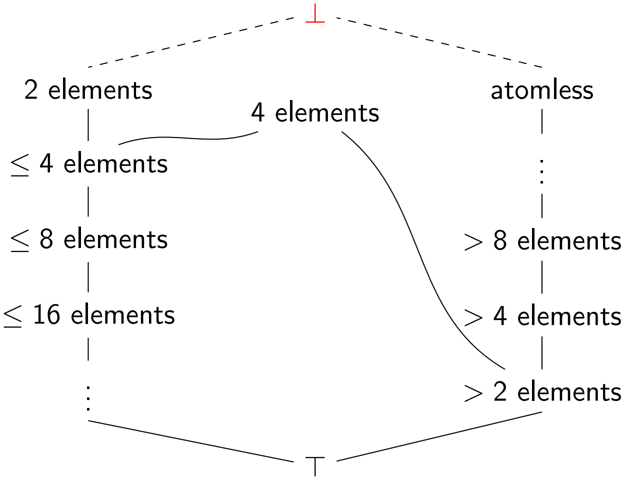
Degrees of parallelism, round 2



Degrees of parallelism, round 2



Degrees of parallelism, round 2



Degrees of parallelism, round 2