PRESERVING CARDINALS AND WEAK FORMS OF ZORN'S LEMMA IN REALIZABILITY MODELS

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Abstract. We develop a technique for representing and preserving cardinals in realizability models, and we apply this technique to define a realizability model of Zorn's Lemma restricted to an ordinal.

1. Introduction

Realizability was introduced by Kleene in 1945 as an attempt to extract the computational content of constructive proofs. The general goal of realizability is to interpret the formulas as sets of programs in a way that the *realizers* of a given formula (namely the programs that are associated to the formula) provide information about the proof of the formula in the considered system or theory. The Curry-Howard correspondence between proofs in intuitionistic logic and programs (interpreted as simply typed lambda-terms) can be regarded as the continuation of Kleene's work.

Research in realizability has later evolved to include classical logic, and in [4] [5] [7] J.L. Krivine developed a technique to extend the proofs-programs correspondence to Zermelo-Fraenkel set theory ZF; we will refer to Krivine's method as *classical realizability*. Krivine's technique generalizes the method of Forcing, which is the main tool in set theory for building models of ZF or ZFC (i.e. Zermelo-Fraenkel set theory plus the Axiom of Choice) and prove relative consistency results. Thus, from a purely mathematical point of view, Forcing models are special cases of classical realizability models, nevertheless these models have no computational content as they involve just one realizer (the maximal condition). We will refer to Forcing models as *trivial realizability models*.

When working with forcing, one starts by assuming the consistency of ZF (respectively ZFC) and considers a model of that theory which is called the *ground model*; then a new model of ZF (respectively ZFC) is built which is a proper extension of the ground model. Unlike forcing models, non-trivial realizability models are not proper extensions of the ground model, in particular they do not have the same ordinals. In this paper, we illustrate a technique for representing and preserving the cardinals of the ground model inside the realizability model. We prove that for every ordinal α in $\mathcal{M}$, we can define a realizability model where α has a representative $\widehat{\alpha}$ such that if α is a cardinal, then $\widehat{\alpha}$ is still a cardinal in the realizability model.

We apply our technique for realizing a version of Zorn's Lemma "restricted to an ordinal α" which we denote by ZL_α (the formal definition will be discussed in Subsection 4.1). The Axiom of Choice, AC, can be proven to be equivalent to the statement "for every ordinal α, ZL_α". Assuming the consistency of ZF plus the Axiom of Global Choice (see Subsection 2.4 for a definition of the Axiom of Global Choice), we consider a model $\mathcal{M}$ of ZF with a global choice function, and we prove that for every ordinal α in $\mathcal{M}$, we can define a realizability model where $\mathrm{ZL}_{\widehat{\alpha}}$ holds for the representative $\widehat{\alpha}$ of α.

This paper is structured as follows. In Section 2, we recall the main notions of classical realizability. In Section 3, we construct for each infinite cardinal κ of the ground model, a realizability model in which all ordinals up to κ have a representative $\hat{\kappa}$ (i.e. they have a well-behaved *name*) In Section 4, we prove that in such a realizability model, Zorn's lemma restricted to $\widehat{\kappa}$ is true. Finally, in Section 5 we show how to make sure that, if κ is a cardinal, then so is $\widehat{\kappa}$.

2. Preliminaries and notation

We assume that the reader is familiar with Zermelo–Fraenkel set theory, with and without the axiom of choice (ZFC and ZF, respectively). In this section we recall the basic notions of classical realizability; our presentation will be slightly different from Krivine's [4, 5, 7].

2.1. **Realizability algebras.** The intuition behind classical realizability as presented in [4] [5] and [7] is that we can use λ_c-calculus to evaluate the *truth value* and the *falsity value* of any formula of classical logic (or even set theory), where λ_c-terms act as truth witnesses and stacks act as falsity witnesses. Truth values and falsity values are related to each other, namely a λ_c-term t is in the truth value of a certain formula if t is "incompatible" with every stack π in the falsity value of the formula (*i.e.* if the process $t * \pi$ is in a certain set $\perp\!\!\!\perp$ called "the pole"). We choose some privileged λ_c-terms that we will call *realizers*, and we show that under some technical conditions (see Subsection 2.5), the set of formulas that are realized by some realizer (that is the formulas whose truth value contain at least a realizer) forms a consistent theory; a realizability model is a model of such a theory.[1]

The main ingredients for constructing a realizability model are the following:

- given a pair of cardinals (κ, μ), we let $\Lambda_{(\kappa,\mu)}$ and $\Pi_{(\kappa,\mu)}$ denote respectively the set of *closed* λ_c*-terms* (or simply, *terms*) and the set of *stacks* as defined by the following grammars, modulo α-equivalence:

λ_c-terms	t, u ::=	x	(variable)
	|	tu	(application)
	|	$\lambda x.t$	(abstraction, where x is a variable and t is a λ_c-term)
	|	cc	(call-with-current-continuation)
	|	k_π	(continuation constant, where π is a stack)
	|	ξ_α	(special instructions, where $\alpha < \kappa$)
Stacks	π ::=	ω_α	(stacks bottoms, where $\alpha < \mu$)
	|	$t \bullet \pi$	(where t is a closed λ_c-term and π is a stack)

As usual we say that a variable x occurs *freely* in given λ_c-term if it occurs outside the scope of an abstraction. The special instructions and the stacks bottoms

[1]The forcing technique is analogous: roughly speaking, we evaluate the truth value of formulas via the elements of a Boolean algebra; a condition p forces φ (namely p realizes φ) if it is incompatible with every condition that forces $\neg\varphi$; the set of formulas that are forced by $\mathbf{1}$ forms a consistent theory, and any model of such a theory is a forcing model.

are customisable constants. When there is no ambiguity we will omit the index (λ, μ) and simply write Λ and Π instead of $\Lambda_{(\kappa,\mu)}$ and $\Pi_{(\kappa,\mu)}$. Application is left associative, thus the term $(\ldots((tu_1)u_2)\ldots)u_n$ will be written $tu_1u_2\ldots u_n$. Application has higher priority than abstraction, thus the term $\lambda x.(tu)$ will be written $\lambda x.tu$

- $\mathcal{R}_{(\kappa,\mu)}$, denotes the set of *realizers*[2], namely the closed λ_c-terms that contain no occurrence of a continuation constant.

- $\Lambda_{(\kappa,\mu)} * \Pi_{(\kappa,\mu)}$ is the set of *processes* defined by the following grammar, modulo α-equivalence:

  Processes $p \ ::= \ t * \pi$ (where t is a closed λ_c-term and π is a stack)

- The *execution* $\prec_K$ which is the smallest preorder on the set of processes such that

$$\begin{array}{lcll} tu * \pi & \succ_K & t * u \bullet \pi & \text{(push)} \\ \lambda x.t * u \bullet \pi & \succ_K & t[x := u] * \pi & \text{(grab)} \\ cc * t \bullet \pi & \succ_K & t * k_\pi \bullet \pi & \text{(save)} \\ k_{\pi'} * t \bullet \pi & \succ_K & t * \pi' & \text{(restore)} \end{array}$$

Note that there is no evaluation rule for the special instructions, thus $\prec_K$ treats the special instructions as inert constants; depending on the context we may define other evaluation relations with specific evaluation rules for the special instructions.

The cardinality of $\Lambda_{(\lambda,\mu)}$, $\Pi_{(\lambda,\mu)}$, $\mathcal{R}_{(\lambda,\mu)}$ and $\Lambda_{(\lambda,\mu)} * \Pi_{(\lambda,\mu)}$ is the maximum of λ, μ and $\aleph_0$.

A *realizability algebra* is a tuple $\mathcal{A} = (\kappa, \mu, \prec, \perp\!\!\!\perp)$, such that:

- κ and μ are cardinals (they fix the number of special instructions and stacks bottoms)
- $\prec$ is a preorder on the set of processes $\Lambda_{(\kappa,\mu)} * \Pi_{(\kappa,\mu)}$ that extends $\prec_K$
- $\perp\!\!\!\perp$ is a final segment of the set of processes, *i.e.* if $t * \pi \succ t' * \pi'$ and $t' * \pi' \in \perp\!\!\!\perp$, then $t * \pi \in \perp\!\!\!\perp$. It is called the *pole* of the realizability algebra.

We assume that $\mathcal{A}$ lives in a model $\mathcal{M}$ of ZFC. $\mathcal{M}$ is called the *ground model.*

2.2. **Realizability algebras and Forcing.** We mentioned in the introduction that classical realizability generalizes forcing. We briefly explain here how we can define a realizability algebra from a forcing boolean algebra – if the reader is not interested in the connection between classical realizability and forcing, they can skip this part and go directly to Subsection 2.3 –.

Roughly speaking, a forcing notion $\mathbb{B}$ corresponds to a special case of realizability algebra where the terms and the stack bottoms correspond to the elements of $\mathbb{B}$; the constant cc corresponds to the maximal condition 1; the operations pq, $p \bullet q$ and $p * q$ all correspond to $p \wedge q$; each term k_p corresponds to p; the relation $p \succ q$ corresponds to $p \leq q$ (i.e. $p \wedge q = p$) and the only realizer is $\mathbf{1}$. More precisely, given a boolean algebra $\mathbb{B} = (B, 1, 0, \wedge, \vee, \neg)$ we can define a realizability algebra $\mathcal{A}_{\mathbb{B}} = (\kappa, \mu, \prec, \perp\!\!\!\perp)$ as

[2]In [4] [5] and [7], the realizers are called *proof-like terms* and the set of realizers is denoted by $\mathcal{QP}$.

follows: $\kappa := 0$, so that there are no special instructions; μ is the cardinality of $\mathbb{B}$, so there is a stack bottom for every condition of $\mathbb{B}$ (it follows that for every $p \in \mathbb{B}$, we can represent p by the term $k_p \in \Lambda_{(0,\mu)}$). In order to define the preorder and the pole, we first define by induction a function $\tau : \Lambda^*_{(0,\mu)} \cup \Pi_{(0,\mu)} \to \mathbb{B}$ (where $\Lambda^*_{(0,\mu)}$ denotes the set of all – possibly open – λ_c-terms):

- for every stack bottom p, we let $\tau(p) := p$
- for every term t and every stack π, we let $\tau(t \bullet \pi) := \tau(t) \wedge \tau(\pi)$
- for every variable x, $\tau(x) := \tau(cc) := 1$
- for all λ_c-terms t, u we let $\tau(tu) := \tau(t) \wedge \tau(u)$
- for every variable x and every term t, we let $\tau(\lambda x.t) := \tau(t)$
- for every stack π, we let $\tau(k_\pi) := \tau(\pi)$

Then, we let $\prec$ be defined by $t_1 * \pi_1 \succ t_2 * \pi_2$ if and only if $\tau(t_1) \wedge \tau(\pi_1) \leq \tau(t_2) \wedge \tau(\pi_2)$ and we let $\perp\!\!\!\perp$ be the set of all processes $t * \pi$ such that $\tau(t) \wedge \tau(\pi) = 0$.

2.3. **Non extensional set theory.** In order to define a realizability model for set theory, we consider a non-extensional version of set theory, the theory ZF_ε, which we define hereafter. We consider two membership relations, the usual one $\in$ and a non-extensional membership relation ε . For technical reasons that will be explained later in subsection 2.5, we take as primitive the relations $x \notin y$ and $x \not\varepsilon\, y$ (rather than $x \in y$ and $x \mathrel{\varepsilon} y$) as well as $x \subseteq y$. Thus, the language of ZF_ε is a first order language with three binary relation symbols $\not\varepsilon$, $\notin$ and $\subseteq$; formulas are built as usual from atomic formulas (including $\top, \bot$), with only $\Rightarrow$ and $\forall$ ad primitive logical symbols. We shall introduce some abbreviations:

$$
\begin{aligned}
&\neg F :\equiv F \Rightarrow \bot \\
&F \wedge G :\equiv (F \Rightarrow G \Rightarrow \bot) \Rightarrow \bot \\
&F \vee G :\equiv (F \Rightarrow \bot) \Rightarrow (G \Rightarrow \bot) \Rightarrow \bot \\
&F_1, \dots, F_n \Rightarrow F :\equiv F_1 \Rightarrow (\cdots \Rightarrow (F_n \Rightarrow F) \cdots) \\
&\exists x F :\equiv \neg \forall x \neg F \\
&\exists x F :\equiv \neg \forall x \neg F \\
&\exists x \{F_1, \dots, F_n\} :\equiv \neg \forall x (F_1, \dots, F_n \Rightarrow \bot). \\
&x \mathrel{\varepsilon} y :\equiv x \not\varepsilon\, y \Rightarrow \bot \\
&x \in y :\equiv x \notin y \Rightarrow \bot \\
&x = y :\equiv \forall z (x \mathrel{\varepsilon} z \Rightarrow y \mathrel{\varepsilon} z) \\
&x \simeq y :\equiv x \subseteq y \wedge y \subseteq x \\
&\forall x \varepsilon a \; F(x) :\equiv \forall x (x \mathrel{\varepsilon} a \Rightarrow F(x)) \\
&\exists x \varepsilon a \; F(x) :\equiv \neg (\forall x F(x) \Rightarrow x \not\varepsilon\, a)
\end{aligned}
$$

The relation $=$ is the *strong* or *Leibniz equality*, while $\simeq$ is the *weak* or *extensional equality.* The axioms of ZF_ε are the following:

(0) Axioms of extensionality.

$$
\begin{aligned}
&\forall x \forall y [x \in y \iff \exists z \varepsilon y \; (x \simeq z)]; \\
&\forall x \forall y [x \subseteq y \iff \forall z \varepsilon x \; (z \in y)].
\end{aligned}
$$

(1) Axiom schema of foundation.

$$\forall z_1 \dots \forall z_n \; \forall a \; (\forall x \; (\forall y \varepsilon x \; (F(y, z_1, \dots, z_n) \Rightarrow F(x, z_1, \dots, z_n))) \Rightarrow F(a, z_1, \dots, z_n))$$

for every formula $F(x, z_1, \dots, z_n)$.

(2) Axiom schema of comprehension.

$$\forall z_1 \dots \forall z_n \; \forall a \; \exists b \; \forall x \; (x \,\varepsilon\, b \iff (x \,\varepsilon\, a \land F(x, z_1, \dots, z_n)))$$

(for every formula $F(x, z_1, \dots, z_n)$).

(3) Axiom of pairing.

$$\forall a \; \forall b \; \exists x \; (a \,\varepsilon\, x \land b \,\varepsilon\, x)$$

(4) Axiom of union.

$$\forall a \; \exists b \; \forall x \varepsilon a \; \forall y \varepsilon x \; (y \,\varepsilon\, b).$$

(5) Axiom schema of power set.

$$\forall a \; \exists b \; \forall z_1 \dots \forall z_n \; \exists y \varepsilon b \; \forall x \; (x \,\varepsilon\, y \iff (x \,\varepsilon\, a \land F(x, z_1, \dots, z_n)))$$

(for every formula $F(x, z_1, \dots, z_n)$).

(6) Axiom schema of collection.

$$\forall z_1 \dots \forall z_n \; \forall a \; \exists b \; \forall x \varepsilon a \; (\exists y \; F(x, y, z_1, \dots, z_n) \Rightarrow \exists y \varepsilon b \; F(x, y, z_1, \dots, z_n))$$

(for every formula $F(x, y, z_1, \dots, z_n)$).

(7) Axiom schema of infinity.

$$\forall z_1 \dots \forall z_n \; \forall a \; \exists b \; (a \,\varepsilon\, b \land \forall x \varepsilon b \; (\exists y \; F(x, y, z_1, \dots, z_n) \Rightarrow \exists y \varepsilon b \; F(x, y, z_1, \dots, z_n)))$$

(for every formula $F(x, y, z_1, \dots, z_n)$).

Definition 2.1. We say that a formula $F(x, \vec{z})$ is *extensional with respect to x* if ZF_ε proves $\forall \vec{z} \; \forall a, b \; (a \simeq b \Rightarrow F(a, \vec{z}) \Rightarrow F(b, \vec{z}))$.

If we take $\{\notin, \subseteq\}$ as the signature of the language of ZF, then it can be proven that ZF_ε is a conservative extension of ZF (see for instance [5]). Thus, from any model $\mathcal{N}_\varepsilon = (|\mathcal{N}_\varepsilon|, \not\varepsilon, \notin, \subseteq)$ of ZF_ε, by restricting the language and taking the quotient by $\simeq$, we get a model $\mathcal{N} = (|\mathcal{N}_\varepsilon|/\simeq, \in, \subseteq)$ of ZF . Conversely, any model of ZF can be seen as a model of ZF_ε interpreting $\not\varepsilon$ as $\notin$, and $\subseteq$ as usual.

2.4. **The language of realizability.** We assume the consistency of the theory ZF enriched with the Axiom of Global Choice, AGC. The language of ZF + AGC is the language of ZF enriched with a new function symbol $\mathcal{G}$; the theory ZF + AGC is *ZF* extended as follows:

(1) The axiom $\forall x \; (\exists y \; (y \in x) \Rightarrow \mathcal{G}(x) \in x)$ is added,
(2) The Comprehension and Collection schemas are extended to the formulas of the enriched language (*i.e.* to the formulas containing the new function symbol $\mathcal{G}$).

Thus we fix a model $\mathcal{M}$ of ZF + AGC. The model $\mathcal{M}$ is called *the ground model.* We will define and study the realizability model within $\mathcal{M}$. For that, we need to introduce the *language of realizability.* The language of realizability is an extension of the language of ZF_ε to which we add:

- a new constant symbol for each element of $\mathcal{M}$,
- a new function symbol for every *definable class function* f from $\mathcal{M}^n$ to $\mathcal{M}$, that is to say, every function f from $\mathcal{M}^n$ to $\mathcal{M}$ such that there exists a formula $F(x_1, \dots, x_n, y)$ in the language of ZF + AGC enriched with one constant per element of $\mathcal{M}$, such that for all $a_1, \dots, a_n, b \in \mathcal{M}$, $\mathcal{M} \models F(a_1, \dots, a_n, b)$ if and only if $f(a_1, \dots, a_n) = b$,
- two binary connectors $\cap$ and $\cup$ between formulas.

For each closed first-order term a of the language of realizability, the *interpretation* $\mathcal{V}(a)$ of a is the element of $\mathcal{M}$ defined as follows:

- if a is a constant, $\mathcal{V}(a) = a$,
- if a is of the form $f(b_1, \ldots, b_n)$ with f a definable class function from $\mathcal{M}^n$ to $\mathcal{M}$ and $b_1, \ldots, b_n$ closed first-order terms, then $\mathcal{V}(a) = f(\mathcal{V}(b_1), \ldots, \mathcal{V}(b_n))$.

The function $\mathcal{V}$ just defined determines the interpretation of closed terms in the (yet to be defined) realizability model, however we should point out that the intuitive meaning of each function symbol in the realizability model will sometimes turn out to be very different from the meaning of the underlying function in $\mathcal{M}$. By abuse of notation, we will generally denote the interpretation of a by a.

2.5. **Realizability models.** From now until the end of this section, we fix a realizability algebra $\mathcal{A} = (\lambda, \mu, \prec, \perp\!\!\!\perp)$ in $\mathcal{M}$. To simplify the notation we will write Λ and Π for $\Lambda_{(\lambda,\mu)}$ and $\Pi_{(\lambda,\mu)}$ respectively. For each closed formula φ in the language of realizability, we define a *truth value* $|\varphi| \subseteq \Lambda$ and a *falsity value* $\|\varphi\| \subseteq \Pi$ by induction on the length of the formula (recall that we took $a \not\varepsilon b$ and $a \notin b$ as primitive relations).

Definition 2.2.
- $|\varphi| = \{t \in \Lambda;\ \forall \pi \in \|\varphi\|\ (t * \pi \in \perp\!\!\!\perp)\}$
- $\|\top\| = \emptyset$, $\|\bot\| = \Pi$,
- $\|a \not\varepsilon b\| = \{\pi \in \Pi;\ (a, \pi) \in b\}$ (by which we mean $\{\pi \in \Pi;\ (\mathcal{V}(a), \pi) \in \mathcal{V}(b)\}$),
- $\|a \subseteq b\|$ and $\|a \notin b\|$ are defined simultaneously by induction on $(\mathrm{rk}(a) \cup \mathrm{rk}(b), \mathrm{rk}(a) \cap \mathrm{rk}(b))$ ($\mathrm{rk}(a)$ being the rank of a):
  - $\|a \subseteq b\| = \{t \bullet \pi;\ (t, \pi) \in \Lambda \times \Pi,\ (c, \pi) \in a \text{ and } t \in |c \notin b|\}$
  - $\|a \notin b\| = \{t \bullet t' \bullet \pi;\ (t, t', \pi) \in \Lambda \times \Lambda \times \Pi,\ (c, \pi) \in b,\ t \in |a \subseteq c|,\ t' \in |c \subseteq a|\}$,
- $\|A \Rightarrow B\| = \{t \bullet \pi; (t, \pi) \in \Lambda \times \Pi,\ t \in |A|, \pi \in \|B\|\}$,
- $\|\forall x A\| = \{\pi \in \Pi;\ \exists a(\pi \in \|A[a/x]\|)\}$,
- $\|A \cap B\| = \|A\| \cup \|B\|$,
- $\|A \cup B\| = \|A\| \cap \|B\|$.

We write $t \Vdash \varphi$ for $t \in |\varphi|$.

We can think of $\|\varphi\|$ as the set of all the stacks 'witnessing' the falsity of φ. In particular, no stack can witness that $\top$ is false, whereas every stack can witness that $\bot$ is false, so we have $\|\top\| = \emptyset$ and $\|\bot\| = \Pi$. The falsity value of $a \not\varepsilon b$ contains all the stacks π such that $(a, \pi) \in b$, namely all the stacks that '*witness*' that a belongs to b; at this point it should be clear why we chose to take $a \not\varepsilon b$ and $a \notin b$ as atomic formulas, in fact it is easier to define what constitutes a 'witness' of the falsity of $a \not\varepsilon b$ or $a \notin b$ rather than a 'witness' of the falsity of $a\ \varepsilon\ b$ or $a \in b$. The definition of the falsity value for the other formulas can be justified by similar arguments by considering the axioms of ZF_ε. On the other hand, a realizer of a certain formula is a term which is somehow 'incompatible' with every stack which witnesses the falsity of the formula, namely the realizer and the stack form a process which belongs to the pole.

We say that a formula φ is *realized* if there exists $t \in \mathcal{R}$ such that $t \Vdash \varphi$. We often omit the realizers and write $\Vdash \varphi$ to say that φ is realized (by some realizer).

The following theorems establish that the set of all closed formulas that are realized forms a (classical) theory (*i.e.* it contains classical tautologies and it is closed by modus ponens) which extends ZF_ε. In addition, they give a necessary and sufficient condition on $\perp\!\!\!\perp$ for this theory to be consistent.

Theorem 2.3. *(Krivine [5]) (Adequacy lemma) Let $A_1, ..., A_n, A$ be closed formulas of* ZF_ε *and suppose that $x_1 : A_1, ..., x_n : A_n \vdash t : A$ is derivable (with the usual Curry-style typing rules of natural deduction, see for instance [5]). If $\xi_1 \Vdash A_1, ..., \xi_n \Vdash A_n$, then $t[\xi_1 := x_1, ..., \xi_n := x_n] \Vdash A$. In particular, if $\vdash t : A$, then $t \Vdash A$.*

Theorem 2.4. *(Krivine [5]) Peirce's law is realized, namely for all formulas A, B, we have $cc \Vdash ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$.*

Theorem 2.5. *(Krivine [5]) The axioms of* ZF_ε *are realized.*

The realizability algebra $\mathcal{A}$ is said to be *consistent* if for every realizer t, there exists a stack π such that $t * \pi \notin \perp\!\!\!\perp$.

Theorem 2.6. *(Krivine [5]) The realizability algebra $\mathcal{A}$ is* consistent *if and only if the formula $\perp$ is not realized.*

We call *realizability model of* $\mathcal{A}$ and denote by $\mathcal{M}^{\mathcal{A}}$ the function that maps each closed formula A of the language of realizability (which includes in particular all formulas of ZF) to its truth value $|A|$. We say that the realizability model *satisfies* A and write $\mathcal{M}^{\mathcal{A}} \models A$ if $|A| \cap \mathcal{R} \neq \emptyset$, *i.e.* if A is realized. We say that $\mathcal{M}^{\mathcal{A}}$ is *consistent* if it does not satisfy $\perp$ (which amounts to saying that $\mathcal{A}$ is consistent).

Note that the realizability model is neither a Tarski model nor a Boolean model, however one could obtain a Boolean model by taking a quotient by the equivalence relation that equates two truth values X and Y if and only if $X \Leftrightarrow Y$ is realized. Alternatively, if $\perp$ is not realized, one could obtain a Tarski model by completeness.

Notation 2.7. Given two first order terms a, b and A a formula of the language of realizability, we write

- $a \neq b$ for the formula $a \notin f(b)$, where f is the function from $\mathcal{M}$ to $\mathcal{M}$ defined by $f(x) = \{x\} \times \Pi$,
- $a = b \hookrightarrow A$ for the formula $(a \neq b) \cup A$.

It follows from Definition 2.2 that

- $\|a \neq b\| := \begin{cases} \emptyset & \text{if } a \neq b \\ \Lambda & \text{otherwise,} \end{cases}$
- $\|a = b \hookrightarrow A\| := \begin{cases} \emptyset & \text{if } a \neq b \\ \|F\| & \text{otherwise.} \end{cases}$

It is proved in [5] that the following equivalence is realized:

$$\forall x, y, z_1, ..., z_n((x = y \hookrightarrow A(x, y, z_1, ..., z_n)) \iff (x = y \Rightarrow A(x, y, z_1, ..., z_n))).$$

Yet, $a = b \hookrightarrow A$ is simpler to realize.

2.6. **Special function symbols.** We defined the language of realizability in such a way that every function or class function definable in $\mathcal{M}$ is a first order term of the language. In particular, we will make use of the following class functions.

- pair is an arbitrary definable injective class function from $\mathcal{M}^2$ to $\mathcal{M}$. We can take, for instance, the usual pair function, but we should point out that in the realizability model, the formula $\forall a, b\ (\mathrm{pair}(a, b) = \{\{a\}, \{a, b\}\})$ does not hold anymore. Nevertheless, (no matter what injective class function one chooses) pair can still be used as a pairing construction, since the equivalence

$$\forall x, x', y, y'\ (\mathrm{pair}(x, y) = \mathrm{pair}(x', y') \iff x = x' \wedge y = y')$$

is realized (it is easy to check).

- img is defined by $\text{img}(f) := \{(y,\pi);\ \exists x\ (\text{pair}(x,y),\pi) \in f\}$ for every $f \in \mathcal{M}$. It is easy to check that the formula $\forall f \forall y\ (y\ \varepsilon\ \text{img}(f) \iff \exists x\ \ \text{pair}(x,y)\ \varepsilon\ f)$ is realized.

Another important function symbol is *Gimel*, which is the function $\gimel : \mathcal{M} \to \mathcal{M}$ defined by $\gimel(a) = a \times \Pi$ (see [5] and [7]). In the language of realizabity, the formula $a\ \varepsilon\ \gimel A$ may be understood as "a is of type A": for instance, $\gimel\mathbb{N}$ may be considered as the type of the integers (whose elements may actually not all be integers). The logical operators $\wedge, \vee, \neg$ are functions with domains $\{0,1\} \times \{0,1\}$ and $\{0,1\}$, and take values in $\{0,1\}$; they define in the realizability model a structure of Boolean algebra on $\gimel 2$, which is called the *characteristic Boolean algebra*. We should point out that in the realizability models, $\gimel 2$ may be different than $\{0,1\}$ (see [5]), in that case one can conclude that the realizability model is not a forcing model [7].

Following Krivine [6], for all formulas $F(x_1,\dots,x_n)$, we define a function $(a_1,\dots,a_n) \mapsto \langle F(a_1,\dots,a_n)\rangle$ from $\mathcal{M}^n$ to $\mathcal{M}$ by

$$\langle F(a_1,\dots,a_n)\rangle = \begin{cases} 1 \text{ if } \mathcal{M} \models F(a_1,\dots,a_n) \\ 0 \text{ otherwise.} \end{cases}$$

in the realizability model, the function symbol $\langle F(x_1,...,x_n)\rangle$ takes values in $\gimel 2$ and if a binary relation $x \prec y$ is well founded in $\mathcal{M}$, then the relation $\langle x \prec y\rangle = 1$ is well founded in the realizability model (see [6, Theorem 8]).

A useful property of forcing is that every set in a forcing model has a "name", that is there is a (first order) term in the language of forcing that is interpreted as this set in the forcing model. On the contrary, the elements of a realizability model do not necessarily have a "name", namely in general, for a given set a in a realizability model there is no first order term of the language of realizability that is interpreted as a. Nevertheless, for every definable element x of the realizability model, the singleton $\{x\}$ has a name, as stated in the following lemma.

Lemma 2.8. *(Naming of singletons) For every formula $A(w_1,...,w_n,x)$ in the language of realizability, we can define a class function s_A from $\mathcal{M}^n$ to $\mathcal{M}$ such that the following formulas are realized:*

(1) $\forall \vec{a} \forall x (x\ \varepsilon\ s_A(\vec{a}) \Rightarrow A(\vec{a},x))$

(2) $\forall \vec{a} (\exists x A(\vec{a},x) \Rightarrow \exists x (x\ \varepsilon\ s_A(\vec{a})))$

In other words:

(1) the set $s_A(\vec{a})$ is included into the class $\{x;\ A(\vec{a},x)\}$,

(2) the set $s_A(\vec{a})$ is empty only when the class $\{x : A(\vec{a},x)\}$ is,

in particular, when the class $\{x;\ A(\vec{a},x)\}$ is a singleton, $s_A(\vec{a})$ denotes that singleton.

Proof. We work in $\mathcal{M}$. By using the global choice function g, we can define a (class) function $f : \mathcal{M}^n \times \Lambda \to \mathcal{M}$ such that for every $\vec{a}$ in $\mathcal{M}^n$ and every term $t \in \Lambda$, the set $f(\vec{a},t)$ is an element in the class $\{x \in \mathcal{M};\ t \Vdash A(\vec{a},x)\}$ unless this class is empty (in which case $f(\vec{a},t)$ is any set). We let

$$s_A(\vec{a}) := \{(f(\vec{a},t), t \bullet \pi);\ \pi \in \Pi, t \in \Lambda \text{ and } f(\vec{a},t) \text{ is defined}\}.$$

Then, for every set $b \in \mathcal{M}$, we have $\|\neg A(\vec{a},b)\| \supseteq \|b \not\varepsilon\ s_A(\vec{a})\|$. It follows that the identity realizes $\forall \vec{w} \forall x ((A(\vec{w},x) \Rightarrow \bot) \Rightarrow x \not\varepsilon\ s_A(\vec{w}))$, that proves the first claim. Moreover, $\|\forall x (x \not\varepsilon\ s_A(\vec{a}))\| = \|\forall x (A(\vec{a},x) \Rightarrow \bot)\|$, thus the identity realizes $\forall \vec{w} \forall x (x \not\varepsilon\ s_A(\vec{w}) \Rightarrow (A(\vec{w},x) \Rightarrow \bot))$, that proves the second claim. $\square$

3. Representing the ordinals in the realizability models

3.1. **Naming ordinals in the realizability models.** From now on, we fix a model $\mathcal{M}$ of ZF plus the axiom of global choice, which will be our ground model, and an infinite cardinal κ in $\mathcal{M}$. In this section, we define a realizability model $\mathcal{M}(\kappa)$, and for all $\alpha \leq \kappa$, we define a set $\widehat{\alpha}$ that names an ordinal in this model. In subsection 3.2, we will show how to make sure that, as an ordinal, $\widehat{\alpha}$ actually behaves like α.

As mentioned before, we write Λ and Π for $\Lambda_{(\kappa,1)}$ and $\Pi_{(\kappa,1)}$ respectively. The set Λ of all closed terms has cardinality κ: we fix an enumeration $(\nu_\alpha)_{\alpha<\kappa}$ of Λ.

Let χ be a special instruction (*e.g.* the first special instruction ξ_0).

Definition 3.1. We define $\succ$ as the smallest pre-order on $\Lambda * \Pi$ which extends $\succ_K$ and such that for every pair of ordinals $\alpha, \beta < \kappa$, all terms t, u, v and every stack π:

1. if $\alpha < \beta$, then $\chi * \nu_\alpha \bullet \nu_\beta \bullet t \bullet u \bullet v \bullet \pi \succ t \bullet \pi$
2. if $\alpha = \beta$, then $\chi * \nu_\alpha \bullet \nu_\beta \bullet t \bullet u \bullet v \bullet \pi \succ u \bullet \pi$
3. if $\alpha > \beta$, then $\chi * \nu_\alpha \bullet \nu_\beta \bullet t \bullet u \bullet v \bullet \pi \succ v \bullet \pi$

From now on, we fix a final segment $\perp\!\!\!\perp$ of the set of processes with respect to the pre-order $\prec$.

We denote by $\mathcal{A}_\kappa^{\perp\!\!\!\perp}$ the realizability algebra $(\kappa, 1, \prec, \perp\!\!\!\perp)$ and by $\mathcal{M}(\kappa, \perp\!\!\!\perp)$ or simply $\mathcal{M}(\kappa)$ the corresponding realizability model $\mathcal{M}^{\mathcal{A}_\kappa^{\perp\!\!\!\perp}}$. For every ordinal $\alpha \leq \kappa$, we define $\widehat{\alpha} := \{(\widehat{\beta}, \nu_\beta \bullet \pi);\ \pi \in \Pi, \beta < \alpha\}$. We define a well-ordering on the set $\{\widehat{\alpha};\ \alpha \leq \kappa\}$ as follows: $\widehat{\beta} \leq \widehat{\alpha}$ if and only if $\beta \leq \alpha$.

Lemma 3.2. *Let $\beta < \alpha \leq \kappa$, then $\lambda x.x(\nu_\beta) \Vdash \widehat{\beta}\ \varepsilon\ \widehat{\alpha}$*

Proof. Recall that $\widehat{\beta}\ \varepsilon\ \widehat{\alpha}$ is shorthand for $\widehat{\beta} \notin \widehat{\alpha} \to \bot$. Let $t \in |\widehat{\beta} \notin \widehat{\alpha}|$ and $\pi \in \Pi$. By definition, $\nu_\beta \bullet \pi \in \|\widehat{\beta}\ \varepsilon\ \widehat{\alpha}\|$, therefore $t * \nu_\beta \bullet \pi \in \perp\!\!\!\perp$ and $\lambda x.x(\nu_\beta) * t \bullet \pi \in \perp\!\!\!\perp$. ☐

In the spririt of Notation 2.7, we can define a notation "$a\ \varepsilon\ b \hookrightarrow A$" that is equivalent to "$a\ \varepsilon\ b \hookrightarrow A$" but has much simpler realizers:

Notation 3.3. For every ordinal $\alpha \leq \kappa$, every formula A of the language of realizability and every first order term b, we write $b\ \varepsilon\ \widehat{\alpha} \hookrightarrow A$ for the formula $(b \notin \widehat{\alpha}) \cup (\top \Rightarrow A)$.

Remark 3.4. Let $\alpha \leq \kappa$, let b be a closed first order term and A a closed formula of the language of realizability. If b is of the form $\hat{\beta}$ for an ordinal $\beta < \alpha$, then $\|b\ \varepsilon\ \hat{\alpha} \hookrightarrow A\| = \{\nu_\beta \bullet \pi;\ \pi \in \|A\|\}$, else $\|b\ \varepsilon\ \hat{\alpha} \hookrightarrow A\| = \emptyset$.

Lemma 3.5. *For every $\alpha \leq \kappa$, every first order term $b \equiv b(x_1, \ldots, x_n)$ and every formula $F \equiv F(x_1, \ldots, x_n)$ in the language of realizability,*

$$\Vdash \forall x_1, \ldots, x_n\ \left((b\ \varepsilon\ \widehat{\alpha} \hookrightarrow F) \iff (b\ \varepsilon\ \widehat{\alpha} \Rightarrow F)\right).$$

Proof. Without loss of generality, we can assume that F and b are closed. Let $d := \lambda t.\lambda u.t(\lambda x.xu)$, we show that $d \Vdash (b\ \varepsilon\ \widehat{\alpha} \Rightarrow F) \Rightarrow (b\ \varepsilon\ \widehat{\alpha} \hookrightarrow F)$. If b is not of the form $\hat{\beta}$ for some ordinal $\beta < \alpha$, then $\|b\ \varepsilon\ \hat{\alpha} \hookrightarrow F\| = \|\top\|$, thus the implication above is realized. Suppose then that b is of the form $\hat{\beta}$ for some ordinal $\beta < \alpha$. Then by the remark above, we have $\|b\ \varepsilon\ \hat{\alpha} \hookrightarrow F\| = \{\nu_\beta \bullet \pi;\ \pi \in \|F\|\}$. Let $\xi \in \Lambda$ such that $\xi \Vdash b\ \varepsilon\ \hat{\alpha} \Rightarrow F$ and let $\nu_\beta \bullet \pi \in \|b\ \varepsilon\ \hat{\alpha} \hookrightarrow F\|$, then by Lemma 3.2 we have $\lambda x.x(\nu_\beta) \Vdash \hat{\beta}\ \varepsilon\ \hat{\alpha}$. It follows that $\xi * \lambda x.x(\nu_\beta) \bullet \pi \in \perp\!\!\!\perp$. We have $d * \xi \bullet \nu_\beta \bullet \pi \succ \xi(\lambda x.x(\nu_\beta)) * \pi \succ \xi * (\lambda x.x(\nu_\beta)) \bullet \pi \in \perp\!\!\!\perp$, this completes the proof that $d \Vdash (b\ \varepsilon\ \widehat{\alpha} \Rightarrow F) \Rightarrow (b\ \varepsilon\ \widehat{\alpha} \hookrightarrow F)$.

Let $g := \lambda t.\lambda u.\,\text{cc}(\lambda k.u(\lambda x.k(tx)))$, we show that $g \Vdash (b\ \varepsilon\ \widehat{\alpha} \hookrightarrow F) \Rightarrow (b\ \varepsilon\ \widehat{\alpha} \Rightarrow F)$. Let $t, u \in \Lambda$ and $\pi \in \Pi$ such that $t \Vdash b\ \varepsilon\ \widehat{\alpha} \hookrightarrow F$, $u \Vdash b\ \varepsilon\ \widehat{\alpha}$ and $\pi \in \|F\|$. We have

$g * t \bullet u \bullet \pi \succ u * (\lambda x.k_\pi(tx)) \bullet \pi$ while $u \Vdash b \not\varepsilon \widehat{\alpha} \Rightarrow \perp\!\!\!\perp$, so it is enough to prove that $\lambda x.k_\pi(tx)$ realises $b \not\varepsilon \widehat{\alpha}$. If $b = \widehat{\beta}$ for some $\beta < \alpha$, then for every stack π, we have $\lambda x.k_\pi(tx) * \nu_\beta \bullet \pi' \succ t * \nu_\beta \bullet \pi \in \perp\!\!\!\perp$. Otherwise, we have $\|b \not\varepsilon \widehat{\alpha}\| = \emptyset$, which completes the proof. □

Following up on Notation 3.3, we can define "optimized notations" for relativized quantifications over the $\widehat{\alpha}$:

Notation 3.6. For every $\alpha \leq \kappa$ and every formula F, we write $\forall x^{\widehat{\alpha}} F$ for the formula $\forall x (x \,\varepsilon\, \widehat{\alpha} \hookrightarrow F)$ and we write $\exists x^{\widehat{\alpha}} F$ for the formula $\forall x (x \,\varepsilon\, \widehat{\alpha} \hookrightarrow F \Rightarrow \perp) \Rightarrow \perp$.

Notation 3.7. We write $\mathrm{Ord}_\in(a)$ (read "α is an ordinal") for the following formula, which says that a is a transitive set (with respect to $\in$) and that $\in$ is a strict linear order relation on a:

$$\begin{array}{lll} & \forall x{\in}a\ \forall y{\in}x\ y{\in}a & (a \text{ is a transitive set}) \\ \wedge & \forall x{\in}a\ \forall y{\in}a\ \forall z{\in}a\ (x{\in}y \Rightarrow y{\in}z \Rightarrow x{\in}z) & (\in \text{ defines a transitive relation over } a) \\ \wedge & \forall x{\in}a\ \forall y{\in}a\ (x{\in}y \vee x{\simeq}y \vee y{\in}x) & (\in \text{ is a semiconnex relation over } a). \end{array}$$

This amounts to saying that a is an ordinal in the usual sense, since the axiom of foundation guarantees that $\in$ is well-founded.

We can also define a non-extensional version of this formula as follows.

Notation 3.8. We write $\mathrm{Ord}_\varepsilon(a)$ (read "α is a non-extensional ordinal") for the following formula, which says that a is a transitive set (with respect to ε) and that ε is a strict linear order relation on a:

$$\begin{array}{lll} & \forall x\varepsilon a\ \forall y\varepsilon x\ y\varepsilon a & (a \text{ is a transitive set}) \\ \wedge & \forall x\varepsilon a\ \forall y\varepsilon a\ \forall z\varepsilon a\ (x\varepsilon y \Rightarrow y\varepsilon z \Rightarrow x\varepsilon z) & (\varepsilon \text{ defines a transitive relation over } a) \\ \wedge & \forall x\varepsilon a\ \forall y\varepsilon a\ (x\varepsilon y \vee x{=}y \vee y\varepsilon x) & (\varepsilon \text{ is a semiconnex relation over } a). \end{array}$$

It is easier to work with the non-extensional version because the definition of its falsity value is simpler). As it turns out, the non-extensional version implies the extensional one:

Proposition 3.9. $\mathrm{ZF}_\varepsilon \vdash \forall a\ (\mathrm{Ord}_\varepsilon(a) \Rightarrow \mathrm{Ord}_\in(a))$.

Proof. Let us fix a model of ZF_ε. Let a be a non-extensional ordinal (*i.e.* let a be such that $\mathrm{Ord}_\varepsilon(a)$ is true).

Let $x \in a$ and $y \in x$. Let $x' \simeq x$ such that $x' \,\varepsilon\, a$. Then $y \in x'$, so let $y' \simeq y$ such that $y' \,\varepsilon\, x'$. Then $y' \,\varepsilon\, x'$ and $x' \,\varepsilon\, a$, so $y' \,\varepsilon\, a$. Since $y' \simeq y$, we have $y \in a$.

Let $x, y, z \in a$ such that $x \in y$ and $y \in z$. Let $z' \simeq z$ such that $z' \,\varepsilon\, a$. We have $y \in z'$, so let $y' \simeq y$ such that $y' \,\varepsilon\, z'$. We have $x \in y'$, so let $x' \simeq x$ such that $x' \,\varepsilon\, y'$. Then $y' \,\varepsilon\, z'$ and $z' \,\varepsilon\, a$, so $y' \,\varepsilon\, a$. Moreover, $x' \,\varepsilon\, y'$ and $y' \,\varepsilon\, a$, so $x' \,\varepsilon\, a$. Therefore, $x' \in z'$, with $x' \simeq x$ and $z' \simeq z$, which means that $x \in z$.

Let $x, y \in a$. Let $x' \simeq x$ and $y' \simeq y$ such that $x' \,\varepsilon\, a$ and $y' \,\varepsilon\, a$. If $x' \,\varepsilon\, y'$ then $x \in y$, if $x' = y'$ then $x \simeq y$, and if $y' \,\varepsilon\, x'$ then $y \in x$. □

Non-extensional ordinals have a very useful property: they ε-contain a $=$-unique representative of each of their $\in$-elements as shown in the following proposition.

Proposition 3.10. $\mathrm{ZF}_\varepsilon \vdash \forall a\ (\mathrm{Ord}_\varepsilon(a) \Rightarrow \forall x \,\varepsilon\, a\ \forall y \,\varepsilon\, a\ (x \simeq y \Rightarrow x = y))$.

Proof. Let a be a non-extensional ordinal. Let x and y be ε-elements of a such that $x \neq y$. Then one must have $x \,\varepsilon\, y$ or $y \,\varepsilon\, x$. In the fist case, we get $x \in y$, and in the second, $y \in x$. Because of the extensional axiom of well-foundation, in either case, one must have $x \not\simeq y$ □

Clearly, in every model of ZF_ε, the class of all (extensional) ordinals that are equivalent to at least one non-extensional ordinal is a downwards-closed subclass of the ordinals: in other words, this class is either the class of all ordinals or an ordinal itself. At this point, a very natural question is: is it true in every model of ZF_ε that this class contains all ordinals? Or at least, is it necessarily true in $\mathcal{M}(\kappa)$? The authors suspect that the answer is negative in both cases. If if is indeed negative at least in the first case, it would mean that each model of ZF_ε comes with a sort of "characteristic ordinal" (that may in fact be the class of all ordinals, and so not an ordinal *stricto sensu*), and it would be interesting to find out what are the properties of this ordinal (for example, it is easy to check that it must be a limit ordinal) and what it can tell us about the whole model.

Proposition 3.11. *For every $\alpha \leq \kappa$, the formula* $\mathrm{Ord}_\varepsilon(\widehat{\alpha})$ *is realized by a realizer* that does not depend on α.

Proof. First we show that $\widehat{\alpha}$ is transitive, namely we show that $\forall x^{\widehat{\alpha}} \forall y(y \not\varepsilon\, \widehat{\alpha} \Rightarrow y \not\varepsilon\, x)$ is realised by the term $\theta = \lambda t.\lambda u.u$. Let $\beta < \alpha$, $c \in M$, $u \in |y \not\varepsilon\, \widehat{\alpha}|$ and $\pi \in \|c \not\varepsilon\, \widehat{\beta}\|$, we want to show that $\theta * \nu_\beta \bullet u \bullet \pi \in \perp\!\!\!\perp$. We have $\delta * \nu_\beta \bullet u \bullet \pi \succ u * \pi$. Moreover, $\pi \in \|c \not\varepsilon\, \widehat{\beta}\|$, so $\|c \not\varepsilon\, \widehat{\beta}\|$ is not empty. Thus there exists $\gamma < \beta < \alpha$ such that $c = \widehat{\gamma}$, therefore $\|c \not\varepsilon\, \widehat{\beta}\| = \{\nu_\gamma \bullet \pi';\ \pi' \in \Pi\} = \|c \not\varepsilon\, \widehat{\alpha}\|$, hence $u * \pi \in \perp\!\!\!\perp$.

An analogous argument shows that ε defines a strict order on $\widehat{\alpha}$. We show that this order is total. For that, we are going to use the instruction χ. We let

$$\tau := \lambda b.\lambda c.\lambda t.\lambda u.\lambda v.\chi bc(tb)(u)(vc)$$

and we show that $\tau \Vdash \forall x^{\widehat{\alpha}} \forall y^{\widehat{\alpha}}(x \not\varepsilon\, y \Rightarrow x \neq y \Rightarrow y \not\varepsilon\, x \Rightarrow \perp)$. Let $\beta, \gamma < \alpha$, $t \in |\widehat{\beta} \not\varepsilon\, \widehat{\alpha}|$, $u \in |\widehat{\beta} \neq \widehat{\alpha}|$, $v \in |\widehat{\alpha} \not\varepsilon\, \widehat{\beta}|$ and $\pi \in \Pi$. If $\beta < \gamma$, then $\tau * \nu_\beta \bullet \nu_\gamma \bullet t \bullet u \bullet v \bullet \pi \succ t * \nu_\beta \bullet \pi \in \perp\!\!\!\perp$, because $t \Vdash (\widehat{\beta} \not\varepsilon\, \gamma) \wedge (\beta < \gamma)$. Thus $\tau * \nu_\beta \bullet \nu_\gamma \bullet t \bullet u \bullet v \bullet \pi \in \perp\!\!\!\perp$. The proof is analogous if $\gamma < \beta$. Suppose $\beta = \gamma$, then $\tau * \nu_\beta \bullet \nu_\gamma \bullet t \bullet u \bullet v \bullet \pi \succ v * \pi \in \perp\!\!\!\perp$, because $v \Vdash (\widehat{\beta} \neq \widehat{\gamma})$ implies $v \Vdash \perp$. ☐

As an immediate consequence of Proposition 3.10 and Proposition 3.9, we get the following result.

Corollary 3.12. *For every $\alpha \leq \kappa$, the formula* $\mathrm{Ord}_\in(\widehat{\alpha})$ *is realized by a realizer* that does not depend on α.

Remark 3.13. It can be proved that there exists a final segment $\perp\!\!\!\perp$ such that $\mathcal{M}(\kappa, \perp\!\!\!\perp)$ is consistent (for example, take $\perp\!\!\!\perp = \emptyset$). Moreover, it can be proved [1] that there exists a final segment $\perp\!\!\!\perp$ such that $\mathcal{M}(\kappa, \perp\!\!\!\perp)$ is consistent and satisfies "$\beth 2$ has more than 2 elements".

We haven't showed yet that $\widehat{\alpha}$ can be regarded as a representative of α. We've seen that for all $\beta < \alpha \leq \kappa$, $\lambda x.x(\nu_\beta)$ realizes $\widehat{\beta}\ \varepsilon\ \widehat{\alpha}$, but ν_β is not always a realizer, so this does not prove that $\widehat{\beta}\ \varepsilon\ \widehat{\alpha}$ holds in the realizability model. Actually, there is no hope of finding a realizer of $\widehat{\beta}\ \varepsilon\ \widehat{\alpha}$ in general: indeed, when ν_β realizes $\perp$ (which can happen if ν_β is not a realizer and $\perp\!\!\!\perp$ is not empty), any such realizer could be turned into a realizer of $\perp$. As we will see in Section 3, the situation gets better if we only try to realize $\widehat{\beta} \in \widehat{\alpha}$, which is what actually matters. We can prove immediately that $\widehat{\alpha}$ increases weakly with α.

Proposition 3.14. *For all $\beta \leq \alpha \leq \kappa$, the formula $\beta \subseteq \alpha$ is realized by a realizer* that does not depend on α and β.

Proof. Since $\mathrm{ZF}_\varepsilon \vdash \forall a, b\ ((\forall c\ (c\ \varepsilon\ a \Rightarrow c\ \varepsilon\ b)) \Rightarrow a \subseteq b)$, it suffices to realize $\forall \gamma\ (\gamma\ \varepsilon\ \widehat{\alpha} \Rightarrow \gamma\ \varepsilon\ \widehat{\beta})$. Since $\alpha \leq \beta$, this formula is realized by the identity $\lambda x.x$. □

3.2. **Representing ordinals in the realizability model.** In this section, we show that, by imposing additional conditions on the pole $\perp\!\!\!\perp$, we can make sure that $\widehat{\alpha}$ actually behaves like α for all $\alpha \leq \kappa$.

With Proposition 3.14, we saw that $\widehat{\alpha}$ increases weakly with α. In order to make sure that it increases strictly, we need to introduce a new special instruction ξ distinct from χ, its role is explained hereafter.

Notation 3.15. If P, Q and $\perp\!\!\!\perp$ are three sets of processes, we write "$P \succ_{\perp\!\!\!\perp} Q$" for "$Q \subseteq \perp\!\!\!\perp$ implies $P \cap \perp\!\!\!\perp \neq \emptyset$".

From now on, in addition to being a final segment, we assume that $\perp\!\!\!\perp$ satisfies the following property: for all $\alpha < \kappa$ and all $t \in \Lambda$,

$$\{\ \xi * t \bullet \pi\ \} \succ_{\perp\!\!\!\perp} \{\ t\nu_{\beta_1}(t\nu_{\beta_2}(\ldots t\nu_{\beta_n} u \ldots)) * \pi;\ \beta_1 \leq \ldots \leq \beta_n < \kappa, \exists i\ \beta_i = \alpha, u \in \Lambda\ \}.$$

The instruction ξ will ensure that successor ordinals below κ will not "collapse" to their predecessors[3].

Let $\Delta\kappa = \{(\widehat{\alpha}, \pi);\ \alpha < \kappa, \pi \in \Pi\}$. By standard arguments (see for example the discussions on the operator ℶ in [5]), for all formulas $F(a)$ of the realizability language, the formula $\forall a \varepsilon \Delta\kappa\ F(a)$ is realized if and only if for all $\alpha < \kappa$, $F(\widehat{\alpha})$ is realized by a realizer *that does not depend on* α. For example, Proposition 3.11 implies that $\mathcal{M}(\kappa)$ satisfies $\forall \alpha\ \varepsilon\ \Delta\kappa\ \mathrm{Ord}_\varepsilon(\alpha)$, Corollary 3.12 implies that $\mathcal{M}(\kappa)$ satisfies $\forall \alpha\ \varepsilon\ \Delta\kappa\ \mathrm{Ord}_\in(\alpha)$, and Proposition 3.14 implies that $\mathcal{M}(\kappa)$ satisfies $\forall \alpha, \beta\ \varepsilon\ \Delta\kappa\ (\langle \beta \leq \alpha \rangle = 1 \Rightarrow \beta \subseteq \alpha)$.

Lemma 3.16. *The realizability model $\mathcal{M}(\kappa)$ satisfies*

$$\forall \alpha \varepsilon \Delta\kappa\ \forall \beta\ (\beta\ \varepsilon\ \alpha \Leftrightarrow \beta\ \varepsilon\ \kappa \wedge \langle \beta < \alpha \rangle = 1).$$

Proof. The term $\lambda x.(\lambda y.y)$ realizes $\forall \alpha \varepsilon \Delta\kappa\ \forall \beta\ (\beta\ \varepsilon\ \alpha \hookrightarrow \langle \beta < \alpha \rangle = 1)$. The identity $\lambda x.x$ realizes $\forall \alpha \varepsilon \Delta\kappa\ \forall \beta\ (\beta\ \varepsilon\ \alpha \Rightarrow \beta\ \varepsilon\ \widehat{\kappa})$. The identity $\lambda x.x$ also realizes $\forall \alpha \varepsilon \Delta\kappa\ \forall \beta\ (\langle \beta < \alpha \rangle = 1 \hookrightarrow \beta\ \varepsilon\ \widehat{\kappa} \Rightarrow \beta\ \varepsilon\ \alpha)$. □

Lemma 3.17. *For all $\alpha \leq \kappa$, the formula $\forall \beta \varepsilon \widehat{\kappa}\ (\beta \in \widehat{\alpha} \Rightarrow \beta\ \varepsilon\ \widehat{\alpha})$ is realized by a realizer that does not depend on α. In other words, $\mathcal{M}(\kappa)$ satisfies*

$$\forall \alpha \varepsilon \Delta\kappa\ \forall \beta \varepsilon \widehat{\kappa}\ (\beta \in \alpha \Rightarrow \beta\ \varepsilon\ \alpha).$$

Proof. Let us reason from within the realizability model. Let $\beta\ \varepsilon\ \widehat{\kappa}$ be such that $\beta \in \widehat{\alpha}$. There exists $\beta' \simeq \beta$ such that $\beta'\ \varepsilon\ \widehat{\alpha}$. We know from the proof of Proposition 3.14 that $\beta'\ \varepsilon\ \widehat{\kappa}$. Since $\widehat{\kappa}$ is a non-extensional ordinal, $\beta = \beta'$, so $\beta\ \varepsilon\ \widehat{\alpha}$. □

For all n-ary functions f from κ^n to κ, let $\widehat{f}$ denote the function from $\{\widehat{\alpha};\ \alpha < \kappa\}^n$ to $\{\widehat{\alpha};\ \alpha < \kappa\}$ defined by $\widehat{f}(\widehat{\alpha_1}, \ldots, \widehat{\alpha_n}) = \widehat{f(\alpha_1, \ldots, \alpha_n)}$. We let s denote the successor function from κ to κ : in particular, for all $\alpha < \kappa$, $\widehat{s}(\widehat{\alpha}) = \widehat{\alpha + 1}$.

Lemma 3.18. *The realizability model $\mathcal{M}(\kappa)$ satisfies*

$$\forall \alpha \varepsilon \Delta\kappa\ \exists \beta \varepsilon \widehat{s}(\alpha)\ \beta \notin \alpha.$$

[3]If the reader is familiar with forcing in set theory, this can be regarded as a weak analogue of the κ-closure, although it only preserves *ordinals properties* below κ, not cardinalities.

Proof. Let $\alpha < \kappa$: we need to find a realizer of $(\forall\beta^{\widehat{\kappa}}\ (\langle\beta < \widehat{\alpha}\rangle \neq 1 \Rightarrow \langle\beta < \widehat{\alpha+1}\rangle \neq 1)) \Rightarrow \bot$ that does not depend on α. Let $t \Vdash \forall\beta^{\widehat{\kappa}}\ (\langle\beta < \widehat{\alpha}\rangle \neq 1 \Rightarrow \langle\beta < \widehat{\alpha+1}\rangle \neq 1)$ and $\pi \in \Pi$: we will prove that $\xi{*}t \bullet \pi \in \perp\!\!\!\perp$. It suffices to prove that for all $\beta_1 \leq \ldots \leq \beta_n < \kappa$, all $i \leq n$ such that $\beta_i = \alpha$ and all $u \in \Lambda$, we have $t\nu_{\beta_1}(t\nu_{\beta_2}(\ldots t\nu_{\beta_n}u\ldots)) * \pi \in \perp\!\!\!\perp$. Since $\beta_i = \alpha$, $\beta_i < \alpha+1$ is true and $\beta_i < \alpha$ is false, so $t \Vdash \widehat{\beta_i}\ \varepsilon\ \widehat{\kappa} \hookrightarrow \top \Rightarrow \bot$, and $t\nu_{\beta_i}(t\nu_{\beta_{i+1}}(\ldots t\nu_{\beta_n}u\ldots)) \Vdash \bot$. In addition, for all $j < i$, $\beta_i < \alpha+1$ is true, so $t \Vdash \widehat{\beta_i}\ \varepsilon\ \widehat{\kappa} \hookrightarrow \bot \Rightarrow \bot$, so $t\nu_{\beta_1}(t\nu_{\beta_2}(\ldots t\nu_{\beta_n}u\ldots)) \Vdash \bot$, and in particular $t\nu_{\beta_1}(t\nu_{\beta_2}(\ldots t\nu_{\beta_n}u\ldots)) * \pi \in \perp\!\!\!\perp$. □

From this, we get that $\widehat{\alpha}$ increases strictly with α:

Corollary 3.19. *The realizability model $\mathcal{M}(\kappa)$ satisfies*

$$\forall\alpha\varepsilon\Delta\kappa\ \alpha \in \widehat{s}(\alpha).$$

Proof. Let us reason from within the realizability model. Let $\alpha\varepsilon\Delta\kappa$. By Proposition 3.14, $\alpha \subseteq \widehat{\kappa}$. Since $\widehat{\kappa}$ is an ordinal, it suffices to prove that $\widehat{s}(\alpha) \not\subseteq \alpha$, which is a consequence of lemmas 3.17 and 3.18. □

Corollary 3.20. *The realizability model $\mathcal{M}(\kappa)$ satisfies*

$$\forall\alpha\varepsilon\Delta\kappa\ \alpha \in \widehat{\kappa}.$$

Corollary 3.21. *The realizability model $\mathcal{M}(\kappa)$ satisfies*

$$\Delta\kappa \simeq \widehat{\kappa}.$$

In fact, $\widehat{s}$ actually computes the successor function in the realizability model (so that, in the realizability model, $\widehat{\alpha+1} = \widehat{s}(\widehat{\alpha})$ is indeed the successor of $\widehat{\alpha}$):

Proposition 3.22. *The realizability model $\mathcal{M}(\kappa)$ satisfies*

$$\forall\alpha\varepsilon\Delta\kappa\ \forall\beta\ (\beta \in \widehat{s}(\alpha) \Leftrightarrow \beta \in \alpha \vee \beta \simeq \alpha).$$

Proof. First, note that for all $\alpha < \kappa$, $\|\forall\beta\ (\beta\ \varepsilon\ \widehat{s}(\alpha) \hookrightarrow \alpha\ \varepsilon\ \beta \hookrightarrow \bot)\| = \emptyset = \|\top\|$.

Let us reason from within the realizability model. The left-to-right implication holds because $\alpha \subseteq \widehat{s}(\alpha)$ and $\alpha \in \widehat{s}(\alpha)$. In addition, by Proposition 3.11, for all $\beta\ \varepsilon\ \widehat{s}(\alpha)$, we have $\beta\ \varepsilon\ \alpha$, $\beta = \alpha$ or $\alpha\ \varepsilon\ \beta$. We cannot have $\alpha\ \varepsilon\ \beta$ by the above remark, so $\beta\ \varepsilon\ \alpha$ or $\beta = \alpha$. □

On the other hand, $\|\forall\alpha^{\widehat{0}}\ \bot\| = \emptyset = \|\top\|$, so $\mathcal{M}(\kappa)$ satisfies "$\widehat{0}$ is the empty set". As a result, for all $n \in \mathbb{N}$, $\mathcal{M}(\kappa)$ satisfies "$\widehat{n}$ is the n-th finite ordinal".

In addition, for all $\alpha < \kappa$, if α is a limit (respectively, successor) ordinal of $\mathcal{M}$, then $\widehat{\alpha}$ is a limit (respectively, successor) ordinal of $\mathcal{M}(\kappa)$:

Proposition 3.23. *Let $\widehat{\mathrm{Limit}}(x)$ (respectively, $\widehat{\mathrm{Successor}}(x)$) denote the formula in the language of $\mathcal{M}$ that says "$x = \widehat{\alpha}$ for some limit (respectively, successor) ordinal $\alpha \leq \kappa$".*

The realizability model $\mathcal{M}(\kappa)$ satisfies "for all $\alpha\ \varepsilon\ \Delta\kappa$, if $\langle\widehat{\mathrm{Limit}}(\alpha)\rangle = 1$, then α is a limit ordinal, and if $\langle\widehat{\mathrm{Successor}}(\alpha)\rangle = 1$, then α is a successor ordinal".

Proof. Let p denote the function from κ to κ that maps α to its predecessor if it has one, and to itself otherwise: it is clear that $\forall\alpha\varepsilon\Delta\kappa\ \widehat{p}(\widehat{s}(\alpha)) = \alpha$ and $\forall\alpha\varepsilon\Delta\kappa\ (\langle\widehat{\mathrm{Successor}}(\alpha)\rangle = 1 \hookrightarrow \widehat{s}(\widehat{p}(\alpha))) = \alpha)$ are realized.

Let us reason from within the realizability model. Assume $\langle\widehat{\mathrm{Limit}}(\alpha)\rangle = 1$. First, $\langle\widehat{1} \leq \alpha\rangle = 1$, so $\alpha \not\simeq \emptyset$.

Let $\beta\ \varepsilon\ \alpha$:

- $\beta \in \widehat{s}(\beta)$
- $\widehat{s}(\beta) \;\varepsilon\; \Delta\kappa$, so $\widehat{s}(\beta) \in \widehat{s}(\widehat{s}(\beta))$
- $\langle \widehat{s}(\widehat{s}(\beta)) \leq \alpha \rangle = 1$, so $\widehat{s}(\widehat{s}(\beta)) \subseteq \alpha$,

so for all $\beta \;\varepsilon\; \alpha$, $\widehat{s}(\beta) \in \alpha$. Since $\widehat{s}(\beta)$ is the successor ordinal of β, α is limit.

Now, assume that $\langle \widehat{\text{Successor}}(\alpha) \rangle = 1$: then $\widehat{s}(\widehat{p}(\alpha))) = \alpha$, and $\widehat{s}(\widehat{p}(\alpha)))$ is the successor of $\widehat{p}(\alpha)$. □

This means that in $\mathcal{M}(\kappa)$, $\widehat{\omega}$ is at least the first limit ordinal (*i.e.* "$\mathcal{M}(\kappa)$'s version of ω"); in fact, they are equal:

Proposition 3.24. *The realizability model $\mathcal{M}(\kappa)$ satisfies "for all $\alpha \in \widehat{\omega}$, $\alpha \simeq \emptyset$ or α is a successor ordinal".*

Proof. The formula $\forall \alpha^{\widehat{\omega}}\ \forall \beta\ (\langle \beta < \alpha \rangle = 1 \hookrightarrow \langle \widehat{\text{Successor}}(\alpha) \rangle = 1)$ is realized by $\lambda x.\lambda y.y$. □

Corollary 3.25. *The realizability model $\mathcal{M}(\kappa)$ satisfies "$\widehat{\omega}$ is the first limit ordinal".*

Remark 3.26. As in Subsection 3.1, letting $\perp\!\!\!\perp = \emptyset$ makes $\mathcal{A}_\kappa^{\perp\!\!\!\perp}$ consistent while satisfying all the required conditions. The authors conjecture that, in this case too, there exists a suitable $\perp\!\!\!\perp$ such that the realizability model satisfies "$\gimel 2$ has more than 2 elements".

4. Zorn's lemma restricted to a single ordinal

4.1. **Zorn's lemma and the Axiom of Choice.** In this section we discuss Zorn's lemma restricted to a single ordinal. Given a binary relation R over a set or the whole universe, and a sequence $s = (s_\beta)_{\beta<\alpha}$ indexed by some ordinal α, we say that s is an *R-increasing sequence* or just an *R-sequence* if for $\beta < \gamma < \alpha$, we have $s_\beta \; R \; s_\gamma$.

Definition 4.1. Given any model of ZF and any ordinal κ in that model, we call *Zorn's lemma restricted to* κ, the following axiom schema: for every formula $R(\vec{w}, x, y)$ (in the language of ZF), for every tuple $\vec{a}$, such that $R(\vec{a}, x, y)$ defines a strict partial ordering relation over the whole model, if every set that is linearly ordered by R has a strict upper bound, then there is an R-increasing sequence indexed by κ. We write ZL_κ for Zorn's lemma restricted to κ.

Zorn's lemma restricted to κ corresponds to a weak version of the contrapositive of Zorn's Lemma (see the proof of Proposition 4.2). For every ordinal κ in the ground model, we are going to prove that *Zorn's lemma restricted to* $\widehat{\kappa}$ is true in the realizability model $\mathcal{M}(\kappa)$ defined in the previous section. We shall first show that AC is equivalent to "for every ordinal κ, ZL_κ".

Proposition 4.2. *In any model of* ZF*, Zorn's lemma is true if and only if for every ordinal κ, Zorn's lemma restricted to κ is true.*

Proof. Assume that for all κ, Zorn's lemma restricted to κ is true. Let X be a set and R a strict partial order on X such that every subset of X that is linearly ordered by R has a non-necessarily strict upper bound. Let κ be the least ordinal such that there is no injective function from κ to X. Then, there cannot be an R-increasing sequence indexed by κ so, by ZL_κ, there must be an R-increasing sequence C with an upper bound y but no strict upper bound: this means that y is maximal.

Now, assume the Axiom of Choice. Suppose that R defines a strict partial order relation over the universe such that every R-increasing sequence has a strict upper bound. We define by induction an R-increasing sequence over the whole class of ordinals

$(a_\alpha)_{\alpha\in\mathrm{Ord}}$ as follows. We let a_0 be any set. Suppose that $(a_\alpha)_{\alpha<\beta}$ is defined for an ordinal β. By hypothesis this sequence has a strict upper bound, with a choice function we can choose an upper bound, and we call it a_β. Then, for every ordinal κ, the restricted sequence $(a_\alpha)_{\alpha<\kappa}$ is an R-increasing sequence indexed by κ as required by Zorn's lemma restricted to κ. □

Zorn's lemma restricted to a single ordinal κ is in fact equivalent to the principle of κ dependent choices, denoted by DC_κ. We recall that the axiom DC_κ is the following statement: "let X be a non-empty set and R a binary relation such that for every $\alpha<\kappa$ and every R-sequence $s=(s_\beta)_{\beta<\alpha}$ of elements of X indexed by α, there exists $y\in X$ such that $s\ R\ y$ (i.e. $s_\beta\ R\ y$ for all $\beta<\alpha$), then there is an R-sequence $s'=(s_\beta)_{\beta<\kappa}$ of elements of X indexed by κ.".

Proposition 4.3. *Given any model of* ZF *and any ordinal κ in that model, Zorn's lemma restricted to κ is true if and only if the principle of κ dependent choices is.*

Proof. First, assume Zorn's lemma restricted to κ. Let X be a non-empty set and R a binary relation such that for every $\alpha<\kappa$ and every α-sequence $s=(s_\beta)_{\beta<\alpha}$ of elements of X, there exists $y\in X$ such that $s\ R\ y$. We assume by contradiction that there is no R-sequence of elements of X indexed by κ. Let Y be the set of all R-sequences of elements of X indexed by ordinals less than κ. Let S be the strict prefix ordering on Y. We show that any S-sequence has a strict upper bound. Let C be any S-sequence, its union $s=(s_\beta)_{\beta<\alpha}$ is an R-sequence indexed by an ordinal $\alpha\le\kappa$. We assumed that there is no R-sequence indexed by κ, so we must have $\alpha<\kappa$ and therefore $s\in Y$. Let s_α be such that $s\ R\ s_\alpha$: then $(s_\beta)_{\beta\le\alpha}$ is a strict upper bound of C (for the S ordering relation). By ZL_κ applied to S, there is an S-increasing sequence indexed by κ; the elements of such a sequence must be in Y, thus its union gives us an R-sequence of elements of X indexed by κ, a contradiction.

Now, assume the principle of κ dependent choices. Let $R(\vec{w},x,y)$ be a formula in the language of ZF. Let $\vec{a}$ be sets such that the class $R(\vec{a},-,-)=\{(x,y);\ R(\vec{a},x,y)\}$ defines a strict partial ordering over the whole model such that every set that is linearly ordered has a strict upper bound. Note that we cannot apply DC_κ directly to the relation $R(\vec{a},-,-)$, because it may be a proper class and DC_κ only applies to relations that are sets. By the Reflection Theorem, we can choose an ordinal μ with cofinality at least κ such that V_μ contains $\vec{a}$ and V_μ reflects the formula $R(\vec{w},x,y)$ (free variables: $\vec{w},x,y$) as well as the formula $\exists y\ \forall x{\in}X\ R(\vec{w},x,y)$ (free variables: $\vec{w},X$). Let R_μ denote the restriction of the relation $R(\vec{a},-,-)$ to the set V_μ. Let $s=(s_\beta)_{\beta<\alpha}$ be an R_μ-sequence of elements of V_μ indexed by some ordinal $\alpha<\kappa$. Since μ has cofinality at least κ, $\{s_\beta;\ \beta<\alpha\}\in V_\mu$. Therefore, by our reflection hypotheses, there exists in V_μ a strict $R(\vec{a},-,-)$-upper bound of $\{s_\beta;\ \beta<\alpha\}$; in other words, there exists $y\in V_\mu$ such that $sR_\mu y$. Therefore, by DC_κ, there exists an R_μ-sequence of elements of V_μ that is indexed by κ. In particular, there exists an $R(\vec{a},-,-)$-sequence that is indexed by κ, as required. □

The axiom of choice is equivalent to "for every ordinal κ, DC_κ" (see for instance [3]). It follows, once again, that AC is equivalent to "for every ordinal κ, ZL_κ".

Now, we will formalize these restricted Zorn's lemmas within the language of realizability.

Notation 4.4. (1) For every formula $R(\vec{w},x,y)$ of the language of realizability, we denote by $\mathrm{Upper}^R(\vec{w},c,m)$ the formula $\forall x{\in}c\ R(\vec{w},x,m)$.

(2) For every formula $R(\vec{w},x,y)$ of ZF_ε, we denote by $\mathrm{Chain}^R(\vec{w},c)$ the formula $\forall x{\in}c\ \forall y{\in}c\ (x \neq y \Rightarrow R(\vec{w},x,y) \vee R(\vec{w},y,x))$

Definition 4.5. Let α be a closed first-order term and $R(\vec{w},x,y)$ and $Z_R(\vec{w},\beta,y)$ two formulas of the language of realizability. We say that Z *is an* (R,α)*-ascent* if the conjunction of the following three formulas is realized:

(1) $\forall\vec{w}\ (\forall c\ (\mathrm{Chain}^R(\vec{w},c) \Rightarrow \exists m\ \ \mathrm{Upper}^R(\vec{w},c,m)) \Rightarrow \forall\beta{\in}\alpha\ \exists y\ Z(\vec{w},\beta,y))$,
(2) $\forall\vec{w}\ \forall\beta{\in}\alpha\ \forall\gamma{\in}\alpha\ \forall y\ \forall z\ (Z(\vec{w},\beta,y) \Rightarrow Z(\vec{w},\gamma,z) \Rightarrow \beta \in \gamma \Rightarrow R(\vec{w},y,z))$,
(3) $\forall\vec{w}\ \forall\beta{\in}\alpha\ \forall\gamma{\in}\alpha\ \forall y\ \forall z\ (Z(\vec{w},\beta,y) \Rightarrow Z(\vec{w},\gamma,z) \Rightarrow \beta \simeq \gamma \Rightarrow y \simeq z)$.

Given a any $\alpha < \kappa$, we want to prove that $\mathcal{M}(\kappa)$ satisfies Zorn's lemma restricted to $\widehat{\alpha}$. Since the language of ZF is a subset of the language of realizability, it will be enough to construct, for all $R(\vec{w},x,y)$ extensional with respect to x and y, a formula $Z^R(\vec{w},\beta,y)$ that is extensional with respect to β and y and such that Z^R is an $(R,\widehat{\alpha})$-ascent. This will be the object of Section 4.3.

4.2. **A non-extensional version of the Axiom of Choice.** In this subsection, we discuss a non-extensional version of the Axiom of Choice, called NEAC. We take $\mathcal{M}(\kappa)$ as in the previous section, and we will we show that it satisfies NEAC.

We write $\mathrm{Func}(f)$ for the formula $\forall x,y,y'(\mathrm{pair}(x,y)\ \varepsilon\ f, \mathrm{pair}(x,y')\ \varepsilon\ f \Rightarrow y = y')$. In other words, $\mathrm{Func}(f)$ means that f is functional in the sense of the strong equality $=$. On the other hand, $\mathrm{Func}(f)$ does not imply that f is compatible with the weak equality $\simeq$, namely, if $(x,y) \in f$ and $(x',y') \in f$, then $x \simeq x'$ does not imply $y \simeq y$.

NEAC is the following statement:

$$\forall z\ \exists f\ ((\forall w\varepsilon f\ w\varepsilon z) \wedge \mathrm{Func}(f) \wedge \forall x,y\ \exists y'(\mathrm{pair}(x,y)\ \varepsilon\ z \Rightarrow \mathrm{pair}(x,y')\ \varepsilon\ f))$$

In other words, NEAC establishes that every binary relation can be refined into a function in the non-extensional sense. It should be clear that the extensional analog of NEAC is the Axiom of Choice: suppose that $\{A_i\}_{i\in I}$ is a family of non empty sets, let R be the binary relation defined by $(x,A_i) \in R$ if and only if $x \in A_i$; if R can be refined into a function f compatible with the extensional equality, then f is a choice function. Neverthless, NEAC is not equivalent to the Axiom of Choice (considered as a formula of ZF_ε) as there are many examples of realizability models where NEAC holds but AC fails (see [4], [5], [7]). We want to show that NEAC holds in the realizability model $\mathcal{M}(\kappa)$.

Proposition 4.6. *For every formula $A(w_1,...,w_n,x)$ of the language of realizability there exists a definable class function g_A from $\mathcal{M}^{n+1}$ to $\mathcal{M}$ such that the following formula is realized:* $\forall\vec{w}(\exists x\ A(\vec{w},x) \Rightarrow \exists a\ \varepsilon\ \widehat{\kappa}\ A(\vec{w},g_A(\vec{w},a)))$

Proof. For every term t, we let $P_t := \{\pi;\ t * t \bullet \pi \notin \perp\!\!\!\perp\}$. For every $\vec{w} \in M$ and every $\alpha < \kappa$ such that $P_{\nu_\alpha} \cap \|\forall x \neg A(\vec{w},x)\| \neq \emptyset$, we fix $x \in M$ such that $P_{\nu_\alpha} \cap \|\neg A(\vec{w},x)\| \neq \emptyset$ and we let $g_A(\vec{w},\widehat{\alpha}) = x$. Then, we extend g_A arbitrarily on M^{n+1}.

We show that $\lambda y.yy \Vdash \forall\vec{w}(\forall a^{\widehat{\kappa}} \neg A(\vec{w},g_A(\vec{w},a)) \Rightarrow \forall x \neg A(\vec{w},x))$. Let $\vec{w}$ be a tuple of sets in $\mathcal{M}$, and let $t \in |\forall a^{\widehat{\kappa}} \neg A(\vec{w},g_A(\vec{w},a))|$ and $\pi \in \|\forall x \neg A(\vec{w},x)\|$, we want to show that $t * t \bullet \pi \in \perp\!\!\!\perp$.

Let β be such that $t = \nu_\beta$. Since $t \Vdash \forall a^{\widehat{\kappa}} \neg A(\vec{w},g_A(\vec{w},a))$, in particular for every $\pi' \in \|\neg A(\vec{w},g_A(\vec{w},\widehat{\beta}))\|$, we have $t * t \bullet \pi' \in \perp\!\!\!\perp$. This means that $P_t \cap \|\neg A(\vec{w},g_A(\vec{w},\widehat{\beta}))\| = \emptyset$ thus $P_t \cap \|\forall x \neg A(\vec{w},x)\| = \emptyset$. Since $\pi \in \|\forall x \neg A(\vec{w},x)\|$, we have $\pi \notin P_t$, thus $t * t \bullet \pi \in \perp\!\!\!\perp$. ☐

From this proposition we can show that NEAC holds in $\mathcal{M}(\kappa)$.

Corollary 4.7. *NEAC holds in $\mathcal{M}(\kappa)$.*

Proof. Given $a \in \mathcal{M}$, we want to find $f \in \mathcal{M}$ such that $(\forall w \varepsilon f \; w \varepsilon a) \wedge \mathrm{Func}(f) \wedge \forall x, y \; \exists y' (\mathrm{pair}(x,y) \; \varepsilon \; a \Rightarrow \mathrm{pair}(x,y') \; \varepsilon \; f)$ is realized by a term that does not depend on a.

By Proposition 4.6, there exists a definable class function g such that

$$\forall x \; (\exists y \; \mathrm{pair}(x,y) \; \varepsilon \; a \Rightarrow \exists \alpha \; \varepsilon \; \widehat{\kappa} \; \mathrm{pair}(x, g(a,x,\alpha)) \; \varepsilon \; a)$$

is realized by a term that does not depend on a.

Using the same method Krivine [5] uses to realize the axiom of comprehension, we can construct $f \in \mathcal{M}$ so that the following formula is realized by a term that does not depend on a:

$$\begin{aligned} \mathrm{pair}(x,y) \; \varepsilon \; f \quad \iff \quad & \mathrm{pair}(x,y) \; \varepsilon \; a \\ & \wedge \exists \alpha \; \varepsilon \; \widehat{\kappa} \left(\begin{array}{l} y = g(a,x,\alpha) \\ \wedge \forall \beta \; \varepsilon \; \widehat{\kappa} \; (\beta < \alpha \Rightarrow \mathrm{pair}(x, g(a,x,\alpha)) \not\varepsilon \; a) \end{array} \right). \end{aligned}$$

Intuitively, this means that $f(x)$ is $g(a,x,\alpha)$ for the least $\alpha < \widehat{\kappa}$ such that $g(a,x,\alpha)$ is well-defined.

Since the formula "$\widehat{\kappa}$ is a non-extensional ordinal" is realized, it is clear that $\mathrm{Func}(f)$ is satisfied, hence f is as required.

□

A similar argument shows that we even get non-extensional choice "for classes":

Corollary 4.8. *For each formula $A(\vec{w}, x)$ of the language of realizability, there exists a formula $A^*(\vec{w}, x)$ such that $\mathcal{M}(\kappa)$ satisfies the following three formulas:*

- $\forall \vec{w} \; \forall x \; (A^*(\vec{w}, x) \Rightarrow A(\vec{w}, x))$,
- $\forall \vec{w} \; \forall x \; \forall y \; (A^*(\vec{w}, x) \Rightarrow A^*(\vec{w}, y) \Rightarrow x = y)$,
- $\forall \vec{w} \; (\exists x \; A(\vec{w}, x) \Rightarrow \exists x \; A^*(\vec{w}, x))$.

Remark 4.9. Any non-extensional function whose domain ε-contains an $=$-unique representative of each one of its $\in$-elements (as non-extensional ordinals do, by Proposition 3.10) is automatically compatible with with extensional equality. As a result, in any model of ZF_ε that satisfies NEAC, the statement AC_α ("any family of non-empty sets indexed by α has a choice function (in the extensional sense)" [3]) holds for any ordinal α that is extensionally equivalent to a non-extensional ordinal. In particular, in $\mathcal{M}(\kappa)$, $\mathrm{AC}_{\widehat{\kappa}}$ holds. The next section takes this idea further in order to obtain a stronger version of this result.

4.3. **Zorn's lemma in realizability models.**

Theorem 4.10. *For every $\alpha \leq \kappa$, $\mathcal{M}(\kappa)$ satisfies Zorn's lemma restricted to $\widehat{\alpha}$.*

Proof. Informally, the proof will proceed as follows: we are given a binary relation R (the proof works regardless of whether R is actually a strict ordering or not) such that every R-chain has a strict upper bound. For each $\alpha \in \widehat{\kappa}$, we must choose an $\simeq$-unique[4] y_α in such a way that y_α increases strictly with α.

The first issue we run into is that a priori, it is not possible to make such choices in a way which is compatible with $\simeq$. Fortunately, since $\widehat{\kappa}$ is a non-extensional ordinal, it ε-contains an $=$-unique representative of each of its $\in$-elements (Proposition 3.10). Therefore, it suffices to choose for each $\alpha \; \varepsilon \; \widehat{\kappa}$ an $=$-unique y_α. Then, for each $\alpha' \in \widehat{\kappa}$, it will suffices to let $y_{\alpha'} = y_\alpha$, where α is the unique ε-element of $\widehat{\kappa}$ such that $\alpha \simeq \alpha'$.

[4]In other words, if $\alpha_1 \in \widehat{\kappa}$ and $\alpha_2 \simeq \alpha_1$, we must have $y_{\alpha_2} \simeq y_{\alpha_1}$.

In order to choose y_α, we will need to proceed by induction over α: if y_β has been chosen for all $\beta \mathrel{\varepsilon} \alpha$, then it suffices to choose y_α as one of the strict upper bounds of $\{y_\beta;\ \beta \mathrel{\varepsilon} \alpha\}$, which is an R-chain. However, the next issue we run into is that ZF_ε does not *a priori* allow this kind of "construction by non-extensional induction": we will therefore have to do this inductive construction for "'outside the model", *i.e.* at the level of names in the ground model. In particular, a key step of the below proof is to be able to name, for each $\alpha \leq \kappa$, a unique representative of the R-chain $\{y_\beta;\ \beta \mathrel{\varepsilon} \alpha\}$: this will be the role of the sets F_α, which will be defined below.

Now for the actual proof. Let $R(\vec{w}, x, y)$ be any formula of the language of realizability that is extensional with respect to x and y. For clarity, we will assume that the list of parameters $\vec{w}$ is empty: the proof in the general case is similar, though much less readable. As stated above, all we need to do is construct a formula $Z^R(\beta, y)$ extensional with respect to β and y and such that Z^R is an $(R, \widehat{\alpha})$-ascent.

Recall that img is the class function from $\mathcal{M}$ to $\mathcal{M}$ defined by $\mathrm{img}(f) := \{(y, \pi);\ \exists x\ (\mathrm{pair}(x, y), \pi) \in f\}$ and that the formula $\forall f \forall y\ (y \mathrel{\varepsilon} \mathrm{img}(f) \iff \exists x\ \mathrm{pair}(x, y) \mathrel{\varepsilon} f)$ is true in $\mathcal{M}(\kappa)$.

Let $\mathrm{UpperImg}(f, m)$ be the formula $\mathrm{Upper}^R(\mathrm{img}(f), m)$.

Let $\mathrm{UpperImg}^*(f, m)$ be the formula given by Corollary 4.8 (non-extensional choice for classes) applied to the formula $\mathrm{UpperImg}(f, m)$. Let μ be the function s_{UpperImg^*} from $\mathcal{M}$ to $\mathcal{M}$ given by Lemma 2.8 (naming of singletons). The following formulas are true in $\mathcal{M}(\kappa)$ (by definition of $\mathrm{Upper}^R(\mathrm{img}(f), m)$) :

- $\forall f\ (\exists m\ \mathrm{Upper}^R(\mathrm{img}(f), m) \Rightarrow \exists m\ (m \mathrel{\varepsilon} \mu(f)))$,
- $\forall f\ \forall m\ (m \mathrel{\varepsilon} \mu(f) \Rightarrow \mathrm{Upper}^R(\mathrm{img}(f), m))$,
- $\forall f\ \forall m\ \forall m'\ (m \mathrel{\varepsilon} \mu(f) \Rightarrow m' \mathrel{\varepsilon} \mu(f) \Rightarrow m = m')$.

Now, by induction, for every $\alpha \leq \kappa$ we define a set $F_\alpha \in \mathcal{M}$:

- $F_{\alpha+1} := F_\alpha \cup \{(\mathrm{pair}(\widehat{\alpha}, y), t \bullet \pi);\ (y, \pi) \in \mu(F_\alpha) \wedge t \Vdash \widehat{\alpha} \mathrel{\varepsilon} \widehat{\kappa}\}$,
- for α limit, $F_\alpha := \bigcup_{\beta<\alpha} F_\beta$.

Let $Z^R(\alpha, y)$ denote the following formula:

$$\exists \alpha' \varepsilon \widehat{\kappa}\ \exists y'\ (\alpha' \simeq \alpha \wedge y' \simeq y \wedge \mathrm{pair}(\alpha', y') \mathrel{\varepsilon} F_\kappa),$$

it is clearly extensional with respect to α and y.

The following facts follow from the definition of F_α :

(a) for every $\alpha \leq \kappa$, the statement $\forall \delta\ \forall y\ (\delta \not\varepsilon\ \alpha \Rightarrow \mathrm{pair}(\delta, y) \not\varepsilon\ F_\alpha)$ is realized (by $\lambda x\ .\lambda t.\ tx$)

(b) for every $\alpha < \kappa$, and every $y \in \mathcal{M}$, we have $\|(\widehat{\alpha}, y) \not\varepsilon\ F_\kappa\| = \|\widehat{\alpha} \mathrel{\varepsilon} \widehat{\kappa} \Rightarrow y \not\varepsilon\ \mu(F_\alpha)\|$

(c) for $\delta < \alpha < \kappa$, and $z \in \mathcal{M}$, we have $\|(\widehat{\delta}, z) \not\varepsilon\ F_\kappa\| \equiv \|\mathrm{pair}(\widehat{\delta}, z) \not\varepsilon\ F_\alpha\|$.

It follows that the following formulas are realized:

(d) $\forall \beta\ \forall y\ (\mathrm{pair}(\beta, y) \mathrel{\varepsilon} F_\kappa \Rightarrow \beta \mathrel{\varepsilon} \widehat{\kappa})$

(e) $\forall \gamma \varepsilon \widehat{\kappa}\ \exists f \begin{pmatrix} & \forall \beta\ \forall y\ (\mathrm{pair}(\beta, y) \mathrel{\varepsilon} f \Rightarrow \beta \mathrel{\varepsilon} \gamma) \\ \wedge & \forall z\ (\mathrm{pair}(\gamma, z) \mathrel{\varepsilon} F_\kappa \iff z \mathrel{\varepsilon} \mu(f)) \\ \wedge & \forall \beta \varepsilon \gamma\ \forall y\ (\mathrm{pair}(\beta, y) \mathrel{\varepsilon} F_\kappa \iff \mathrm{pair}(\beta, y) \mathrel{\varepsilon} f) \end{pmatrix}$

(In (e), for all γ, the corresponding f is simply F_γ.)

We can then prove the following claim.

Claim 4.11. Z^R is an $(R, \widehat{\alpha})$-ascent.

Let us reason from within $\mathcal{M}(\kappa)$ and use the same numbering as in Definition 4.5.

Proof of (3): let y, z be sets and let $\beta, \gamma \in \widehat{\kappa}$ be such that $\beta \simeq \gamma$ and such that $Z_R(\beta, y)$ and $Z_R(\gamma, z)$ are true. Let $\beta' \simeq \beta$, $\gamma' \simeq \gamma$, $y' \simeq y$ and $z' \simeq z$ be such that $\beta' \mathrel{\varepsilon} \widehat{\kappa}$, $\gamma' \mathrel{\varepsilon} \widehat{\kappa}$, $\mathrm{pair}(\beta', y') \mathrel{\varepsilon} F_\kappa$ and $\mathrm{pair}(\gamma', z') \mathrel{\varepsilon} F_\kappa$. Then $\beta' \simeq \beta \simeq \gamma \simeq \gamma'$, $\beta' \mathrel{\varepsilon} \kappa$ and $\gamma' \mathrel{\varepsilon} \kappa$, so $\beta' = \gamma'$. Let f be such that $\forall x\ ((\gamma', x) \mathrel{\varepsilon} F_\kappa \iff x \mathrel{\varepsilon} \mu(f))$ (the existence of such a function f is guaranteed by (e) above). Then $y' \mathrel{\varepsilon} \mu(f)$ and $z' \mathrel{\varepsilon} \mu(f)$, so $y' = z'$ and $y \simeq z$.

Proof of (2): Let y, z be sets and let $\beta, \gamma \in \widehat{\kappa}$ be such that $\beta \in \gamma$ and that $Z_R(\beta, y)$ and $Z_R(\gamma, z)$ are true. Let $\beta' \simeq \beta$, $\gamma' \simeq \gamma$, $y' \simeq y$ and $z' \simeq z$ be such that $\beta' \mathrel{\varepsilon} \widehat{\kappa}$, $\gamma' \mathrel{\varepsilon} \widehat{\kappa}$, $\mathrm{pair}(\beta', y') \mathrel{\varepsilon} F_\kappa$ and $\mathrm{pair}(\gamma', z') \mathrel{\varepsilon} F_\kappa$: then $\beta' \mathrel{\varepsilon} \gamma'$. (Indeed, $\beta' \in \gamma'$, so there exists $\beta'' \mathrel{\varepsilon} \gamma'$ such that $\beta'' \simeq \beta'$. Since $\beta' \mathrel{\varepsilon} \widehat{\kappa}$ and $\beta'' \mathrel{\varepsilon} \widehat{\kappa}$ – because $\widehat{\kappa}$ is a transitive set –, $\beta' = \beta''$, so $\beta' \mathrel{\varepsilon} \gamma'$.) Let f be such that $\mathrm{pair}(\gamma', z') \mathrel{\varepsilon} F_\kappa \iff z' \mathrel{\varepsilon} \mu(f)$ and $\mathrm{pair}(\beta', y') \mathrel{\varepsilon} F_\kappa \iff \mathrm{pair}(\beta', y') \mathrel{\varepsilon} f$. Then on the one hand $z' \mathrel{\varepsilon} \mu(f)$, so $\mathrm{Upper}^R(\mathrm{img}(f), z')$ is true, and on the other hand $\mathrm{pair}(\beta', y') \mathrel{\varepsilon} f$, so $R(y', z')$ is true. $R(y, z)$ is extensional with respect to y and z, thus $R(y, z)$ is true.

Proof of (1): let us assume that $\forall c\ (\mathrm{Chain}^R(c) \Rightarrow \exists m\ \ \mathrm{Upper}^R(c, m))$ holds, and let $\gamma \in \widehat{\kappa}$. Let $\gamma' \mathrel{\varepsilon} \widehat{\kappa}$ be such that $\gamma' \simeq \gamma$. Let f be such that the following formulas hold:

(i) $\forall \beta\ \forall y'\ (\mathrm{pair}(\beta, y') \mathrel{\varepsilon} f \Rightarrow \beta \mathrel{\varepsilon} \gamma')$,
(ii) $\forall z'\ (\mathrm{pair}(\gamma', z') \mathrel{\varepsilon} F_\kappa \iff z' \mathrel{\varepsilon} \mu(f))$,
(iii) $\forall \beta \varepsilon \gamma'\ \forall y'\ (\mathrm{pair}(\beta, y') \mathrel{\varepsilon} F_\kappa \iff \mathrm{pair}(\beta, y') \mathrel{\varepsilon} f)$.

It suffices to prove that $\mathrm{Chain}^R(\mathrm{img}(f))$ is true, because it means that $\exists m\ \ \mathrm{Upper}^R(c, m)$ is also true, which means that $\mu(f)$ is non-empty, and thus (by (ii)) that there exists z' such that $\mathrm{pair}(\gamma', z') \mathrel{\varepsilon} F_\kappa$.

Let $y, z \in \mathrm{img}(f)$ be such that $y \not\simeq z$: we need to prove that $R(y, z)$ or $R(z, y)$ holds. Let $y', z' \mathrel{\varepsilon} \mathrm{img}(F)$ be such that $y' \simeq y$ and $z' \simeq z$. Let β and δ be such that $\mathrm{pair}(\beta, y') \mathrel{\varepsilon} f$ and $\mathrm{pair}(\delta, z') \mathrel{\varepsilon} F$. By (i), $\beta \mathrel{\varepsilon} \gamma'$ and $\delta \mathrel{\varepsilon} \gamma'$. By (iii), $\mathrm{pair}(\beta, y') \mathrel{\varepsilon} F_\kappa$ and $\mathrm{pair}(\delta, z') \mathrel{\varepsilon} F_\kappa$, so $Z_R(\beta, y)$ and $Z_R(\delta, z)$ are true. Therefore, by (3), $\beta \not\simeq \delta$. Since γ' is an ordinal, β and δ are ordinals, so we get $\beta \in \delta$ or $\delta \in \beta$. In the first case $R(y, z)$ holds by (2), in the second case $R(z, y)$ holds.

This completes the proof of the theorem. □

5. Preserving cardinals

So far, we have showed that $\widehat{\kappa}$ is an ordinal. However, even though κ is a cardinal, $\widehat{\kappa}$ is not necessarily one. For instance, if we take for κ the cardinal $\aleph_1$, then $\widehat{\kappa}$ may become a countable ordinal. In the case of forcing models, two crucial properties are often used to solve this problem: chain conditions and closure. Any forcing that has the μ-chain condition for some $\mu \leq \kappa$ or μ-closure for some regular $\mu \geq \kappa$ preserves κ as a cardinal in the forcing model (for more detail see *e.g.* Kunen [8]). In this section, we present a general technique for preserving cardinals in realizability models: we introduce a special instruction that we denote η which will ensure that if κ is a cardinal in the ground model, then $\widehat{\kappa}$ is also a cardinal in the realizability model.

Let η and φ be two special instructions distinct from each other and from χ and ξ. Let $(\gamma_\alpha)_{\alpha<\kappa}$ be a family of special instructions pairwise distinct and different from χ, ξ, η and φ.

We now assume that $\perp\!\!\!\perp$ satisfies the following additional properties:

- for every subset $U \subseteq \kappa$ such that the cardinality of $(\kappa \setminus U)$ is strictly less than κ,
$$\{ \varphi * t \bullet \pi \} \succ_{\perp\!\!\!\perp} \{ t * \gamma_\alpha \bullet \pi;\ \alpha \in U \},$$
- for every infinite set of terms A,
$$\{ \eta * t \bullet a \bullet \pi;\ a \in A \} \succ_{\perp\!\!\!\perp} \{ t * a \bullet b \bullet \pi;\ a, b \in A,\ a \neq b \}.$$

The first condition is required for technical reasons. The second is to be thought of as an analogue of the $\aleph_0$-chain condition. Note that although the μ-chain condition makes a forcing notion uninteresting unless $\mu \geq \aleph_1$, it is not the case for realizability.

We are going to show that in that case, $\widehat{\kappa}$ is not collapsed, that is: there is no (extensional) surjection from an $\in$-element of $\widehat{\kappa}$ to $\widehat{\kappa}$. Since $\widehat{\kappa}$ is an ordinal in the non-extensional sense, it is enough to show that there is no non-extensional surjection from an ε-element of $\widehat{\kappa}$ to $\widehat{\kappa}$.

For every formula $F(\vec{w}, x, y)$ of the language of realizability, we write $\mathrm{FunRel}_F(\vec{w})$ for the formula that says that F is a non-extensional functional binary relation, and we denote by $\mathrm{Surj}_F(\vec{w}, a, b)$ the formula that says that F is surjective from a to b in the sense of the non extensional membership relation. More precisely, $\mathrm{FunRel}_F(\vec{w})$ is the formula $\forall x\ \forall y\ \forall y'\ (F(\vec{w}, x, y) \Rightarrow F(\vec{w}, x, y') \Rightarrow y \neq y' \Rightarrow \perp)$, while $\mathrm{Surj}_F(\vec{w}, a, b)$ is the formula $\forall y(\forall x(F(\vec{w}, x, y) \Rightarrow x \notin a) \Rightarrow y \notin b)$.

Theorem 5.1. *For every formula $F(\vec{w}, x, y)$ in the language of realizability, the formula $\forall \vec{w} \forall a^{\widehat{\kappa}}(\mathrm{FunRel}_F(\vec{w}) \Rightarrow \mathrm{Surj}_F(a, \widehat{\kappa}, \vec{w}) \Rightarrow \perp)$ is realized.*

Proof. We define the following terms:

- $\theta_2 := k(\eta f z(\lambda r.r))$
- $\theta_1 := \lambda z.k(fzz\theta_2)$
- $\theta_0 := \lambda a.\lambda f.\lambda s.\varphi(\lambda b.\, \text{cc}(\lambda k.s\theta_1 b))$

We are going to show that $\theta_0 \Vdash \forall \vec{w}\ \forall \mu^{\widehat{\kappa}}(\mathrm{FunRel}_F(\vec{w}) \Rightarrow \mathrm{Surj}_F(\vec{w}, \mu, \widehat{\kappa}) \Rightarrow \perp)$. In order to simplify the notation, we omit the parameters $\vec{w}$. We fix $\mu < \kappa$, and we let $t, u \in \Lambda$ and $\pi \in \Pi$ such that $t \Vdash \mathrm{FunRel}_F$, $u \Vdash \mathrm{Surj}_F(\widehat{\kappa}, \widehat{\lambda})$. We suppose by contradiction that $\theta_0 * \nu_\mu \bullet t \bullet u \bullet \pi \notin \perp\!\!\!\perp$. We let $v := \lambda b.\, \text{cc}(\lambda k.u\theta_1[f := t, s := u]b)$. To simply the notation we write θ_1 for $\theta_1[f := t, k := k_\pi]$ and θ_2 for $\theta_2[f := t, k := k_\pi]$. We let $U := \{ \alpha < \kappa;\ v * \gamma_\alpha \bullet \pi \in \perp\!\!\!\perp \}$. We have $\varphi * v \bullet \pi \notin \perp\!\!\!\perp$, thus by the condition imposed on $\perp\!\!\!\perp$ for φ, the cardinality of $(\kappa \setminus U)$ is equal to κ. We let $B := \{ \beta;\ \exists \alpha \in \kappa \setminus U\ \nu_\beta = \gamma_\alpha \}$. Then B is a subset of κ with cardinality κ.

For every $\beta \in B$, we have $v * \nu_\beta \bullet \pi \notin \perp\!\!\!\perp$, hence $u * \theta_1 \bullet \nu_\beta \bullet \pi \notin \perp\!\!\!\perp$. We know that $u \Vdash \forall x(F(x, \widehat{\beta}) \Rightarrow x \notin \widehat{\mu}) \Rightarrow \widehat{\beta} \notin \widehat{\kappa}$ and $\nu_\beta \bullet \pi \in \|\widehat{\beta} \notin \widehat{\kappa}\|$, therefore $\theta_1 \not\Vdash \forall x(F(x, \widehat{\beta}) \Rightarrow x \notin \widehat{\mu})$. It follows that there exists $\alpha_\beta < \mu$, $\zeta_\beta \in |F(\widehat{\alpha_\beta}, \widehat{\beta})|$ and $\pi'_\beta \in \Pi$ such that $\theta_1 * \zeta_\beta \bullet \nu_{\alpha_\beta} \bullet \pi'_\beta \notin \perp\!\!\!\perp$. If we let $\theta_{2,\beta} := \theta_2[z := \zeta_\beta]$, this implies that $t * \zeta_\beta \bullet \zeta_\beta \bullet \theta_{2,\beta} \bullet \pi \notin \perp\!\!\!\perp$.

We show that $Z := \{\zeta_\beta;\ \beta \in B\}$ has cardinal κ in the ground model. Suppose by contradiction that Z has cardinal less than κ. For every $\zeta \in Z$, we let $B_\zeta = \{\beta \in B : \zeta_\beta = \zeta\}$. Since B has cardinal $\kappa > \mu$ and $B = \bigcup_{\zeta \in Z} B_\zeta$, the set Z must contain an element ζ such that B_ζ has cardinal bigger than μ. For all $\beta, \beta' \in B_\zeta$ such that $\alpha_{\beta'} = \alpha_\beta$, we have $\zeta_\beta = \zeta = \zeta_{\beta'}$ so $\zeta_\beta \Vdash F(\widehat{\alpha_\beta}, \widehat{\beta})$, $\zeta_\beta \Vdash F(\widehat{\alpha_\beta}, \widehat{\beta'})$ and $t \Vdash F(\alpha_\beta, \beta) \Rightarrow F(\alpha_\beta, \beta') \Rightarrow \beta \neq \beta' \Rightarrow \perp$; moreover, we have $t * \zeta_\beta \bullet \zeta_\beta \bullet \theta_{2,\beta} \bullet \pi \notin \perp\!\!\!\perp$, thus $\theta_{2,\beta} \notin |\beta \neq \beta'|$, which implies that $\beta = \beta'$. Hence $\beta \mapsto \alpha_\beta$ defines in the ground model an injection from B_ζ to μ, contradiction.

Since Z has cardinal κ, there is a set $B_0 \subseteq B$ of cardinal κ such that for every $\beta, \beta' \in B_0$, if $\beta \neq \beta'$, then $\zeta_\beta \neq \zeta_{\beta'}$. For every $\beta \in B_0$, we have $t * \zeta_\beta \bullet \zeta_\beta \bullet \theta_{2,\beta} \bullet \pi \notin \perp\!\!\!\perp$, while $t \Vdash F(\widehat{\alpha_\beta}, \widehat{\beta}) \Rightarrow F(\widehat{\alpha_\beta}, \widehat{\beta}) \Rightarrow \widehat{\beta} \neq \widehat{\beta} \Rightarrow \perp\!\!\!\perp$ and $\zeta_\beta \Vdash F(\widehat{\alpha}, \widehat{\beta})$. Therefore, $\theta_{2,\beta} \not\Vdash \widehat{\beta} \neq \widehat{\beta}$. It follows that there exists a stack π''_β such that $\theta_{2,\beta} * \pi''_\beta \notin \perp\!\!\!\perp$, hence $\eta * t \bullet \zeta_\beta \bullet (\lambda r.r) \bullet \pi \notin \perp\!\!\!\perp$.

For every infinite set $B_1 \subseteq B_0$, the set $\{\zeta_\beta : \beta \in B_1\}$ is also infinite, hence by the restriction we imposed on $\perp\!\!\!\perp$ for η, there exists $\beta, \beta' \in B_1$ such that $\zeta_\beta \neq \zeta_{\beta'}$ and $t * \eta_\beta \bullet \eta_{\beta'} \bullet (\lambda r.r) \bullet \pi \notin \perp\!\!\!\perp$. Since $\zeta_\beta \neq \zeta_{\beta'}$, $\beta \neq \beta'$, so $(\lambda r.r)$ realizes $\beta \neq \beta'$. Moreover, we know that $\eta_\beta \Vdash F(\widehat{\alpha_\beta}, \widehat{\beta})$, $\eta_{\beta'} \Vdash F(\widehat{\alpha_{\beta'}}, \widehat{\beta'})$ and $t * \eta_\beta \bullet \eta_{\beta'} \bullet (\lambda r.r) \bullet \pi \notin \perp\!\!\!\perp$. Now, we know that for all $\alpha < \kappa$, $t \Vdash F(\widehat{\alpha}, \widehat{\beta}) \Rightarrow F(\widehat{\alpha}, \widehat{\beta'}) \Rightarrow \widehat{\beta} \neq \widehat{\beta'} \Rightarrow \perp$, so if α_β and $\alpha_{\beta'}$ were equal, we would obtain a contradiction by letting $\alpha := \alpha_\beta = \alpha_{\beta'}$.

In other words, if we say that (α, β) and (α', β') are *incompatible* when $\alpha = \alpha'$ and $\beta \neq \beta'$, then the set $\{(\alpha_\beta, \beta);\ \beta \in B_0\}$ cannot have an infinite antichain (that is an infinite set such that any two distinct elements are incompatible).

It follows that for every $\alpha < \mu$, the set $Y_\alpha := \{\beta \in B_0;\ \alpha = \alpha_\beta\}$ is finite, contradicting the fact that $B_0 = \bigcup_{\alpha<\mu} Y_\alpha$ has cardinality $\kappa > \mu$. □

By combining this theorem with the main theorem of the previous section, we get the final result.

Corollary 5.2. *Let $\mathcal{M}$ be a model of* ZF *with a global choice function and let κ be an infinite cardinal in $\mathcal{M}$, then there is a realizability model where $\widehat{\kappa}$ is a cardinal and* $\mathrm{ZL}_{\widehat{\kappa}}$ *holds.*

Proof. By Theorem 4.10, the algebra $\mathcal{A}$ determines a realizability model where $\mathrm{ZL}_{\widehat{\kappa}}$ holds. Theorem 5.1 ensures that $\widehat{\kappa}$ is a cardinal in this model. □

Remark 5.3. As in Subsections 3.1 and 3.2, letting $\perp\!\!\!\perp = \emptyset$ makes $\mathcal{A}_\kappa^{\perp\!\!\!\perp}$ consistent while satisfying all the required conditions. The authors conjecture that, in this case too, there exists a suitable $\perp\!\!\!\perp$ such that the realizability model satisfies "ℷ2 has more than 2 elements".

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