PRESERVING CARDINALS AND WEAK FORMS OF ZORN'S LEMMA IN REALIZABILITY MODELS

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Abstract. We develop a technique for representing and preserving cardinals in realizability models, and we apply this technique to define a realizability model of Zorn's Lemma restricted to an ordinal.

1. Introduction

Realizability was introduced by Kleene in 1945 as an attempt to extract the computational content of constructive proofs. The general goal of realizability is to interpret the formulas as programs in a way that the *realizers* of a given formula (namely the programs that are associated to the formula) provide information about the proof of the formula in the system or theory considered. Curry-Howard correspondence between proofs in intuitionistic logic and programs (interpreted as simply typed lambda-terms) can be regarded as the continuation of Kleene's work.

Research in realizability has later evolved to include classical logic, and in [3] [4] [6] J.L. Krivine developed a technique to extend the proofs-programs correspondence to Zermelo-Fraenkel set theory ZF; we will refer to Krivine's method as *classical realizability.* Krivine's technique generalizes the method of Forcing, which is the main tool in set theory for building models of ZF or ZFC (i.e. Zermelo-Fraenkel set theory plus the Axiom of Choice) and prove relative consistency results. Thus, from a purely mathematical point of view, Forcing models are special cases of classical realizability models, nevertheless the computational content of these models is void as they involve just one realizer (the maximal condition). We will refer to Forcing models as *trivial realizability models.*

When working with forcing, one starts by assuming the consistency of ZF (respectively ZFC) and considers a model of that theory which is called the *ground model*; then a new model of ZF (respectively ZFC) is built which is a proper extension of the ground model. Unlike forcing models, non-trivial realizability models are not proper extensions of the ground model, in particular they do not have the same ordinals. In this paper, we illustrate a technique for representing and preserving the cardinals of the ground model inside the realizability model. We prove that for every ordinal α in $\mathcal{M}$, we can define a realizability model where α has a representative $\widehat{\alpha}$ such that if α is a cardinal, then $\widehat{\alpha}$ is still a cardinal in the realizability model.

We apply our technique for realizing a version of Zorn's Lemma "restricted to an ordinal α" which we denote by ZL_α: given an order relation R, if every R-chain has a strict upper bound, then there is an R-chain of length α. The Axiom of Choice, AC, can be proven to be equivalent to the statement "for every ordinal α, ZL_α". Assuming the consistency of ZF plus the Axiom of Global Choice, we consider a model $\mathcal{M}$ of

ZF with a global choice function, and we prove that for every ordinal α in $\mathcal{M}$, we can define a realizability model where $\mathrm{ZL}_{\widehat{\alpha}}$ holds for the representative $\widehat{\alpha}$ of α.

This paper is structured as follows. In Section 2, we recall the main notions of classical realizability. In Section 3 we construct for each ordinal κ of the ground model a realizability model in which all ordinals up to κ can be represented (*i.e.* they have a well-behaved *name*). In Section 4, we prove that in such realizability models, Zorn's lemma restricted to κ's representative is true. Finally, in Section 5 we illustrate our technique for preserving cardinals in realizability models.

2. Preliminaries and notation

We assume the reader is familiar with Zermelo–Fraenkel set theory, with and without the axiom of choice (ZFC and ZF, respectively). In this section we recall the basic notions of classical realizability; our presentation will be slightly different from Krivine's [3, 4, 6].

2.1. **Realizability algebras.** The intuition behind classical realizability as presented in [3] [4] and [6] is that we can use λ_c-calculus to evaluate the *truth value* and the *falsity value* of any formula of classical logic (or even set theory), where λ_c-terms act as truth witnesses and stacks act as falsity witnesses. Truth values and falsity values are related to each other, namely a λ_c-term t is in the truth value of a certain formula if t is "incompatible" with every stack π in the falsity value of the formula (*i.e.* if the process $t * \pi$ is in $⫫$). We chose some privileged λ_c-terms that we will call *realizers*[1], and we show that if we respected certain technical constraints, then the set of formulas that are realized by some realizer (that is the formulas whose truth value contain at least a realizer) forms a consistent theory; any model of such a theory is a realizability model.[2]

The main ingredients for constructing a realizability model are the following:

- given a pair of ordinals (κ, μ), we let $\Lambda_{(\kappa,\mu)}$ and $\Pi_{(\kappa,\mu)}$ denote respectively the set of *closed λ_c-terms* (or simply, *terms*) and the set of *stacks* as defined by the following grammars, modulo α-equivalence:

λ_c-terms	t, u	$::=$	x	(variable)
		$\mid$	tu	(application)
		$\mid$	$\lambda x.t$	(abstraction, where x is a variable and t is a λ_c-term)
		$\mid$	cc	(call-with-current-continuation)
		$\mid$	k_π	(continuation constant, where π is a stack)
		$\mid$	ξ_α	(special instructions, where $\alpha < \kappa$)
Stacks	π	$::=$	ω_α	(stacks bottoms, where $\alpha < \mu$)
		$\mid$	$t \bullet \pi$	(where t is a closed λ_c-term and π is a stack)

As usual we say that a variable x occurs *freely* in given λ_c-term if it occurs outside the scope of an abstraction. The special instructions and the stacks bottoms

[1] In Krivine's papers [3] [4] and [6] the realizers are called *proof-like terms*

[2] The forcing technique is analogous: roughly speaking, we evaluate the truth value of formulas via the elements of a Boolean algebra; a condition p forces φ (namely p realizes φ) if it is incompatible with every condition that forces $\neg\varphi$; the set of formulas that are forced by $\mathbf{1}$ forms a consistent theory, and any model of such a theory is a forcing model.

are customisable constants. When there is no ambiguity we will omit the index (λ, μ) and simply write Λ and Π instead of $\Lambda_{(\kappa,\mu)}$ and $\Pi_{(\kappa,\mu)}$. Application is left associative, thus the term $(\ldots((tu_1)u_2)\ldots)u_n$ will be written $tu_1u_2\ldots u_n$. Application has higher priority than abstraction, thus the term $\lambda x.(tu)$ will be written $\lambda x.tu$

- $\mathcal{R}_{(\kappa,\mu)}$, denotes the set of *realizers*[3], namely the closed λ_c-terms that contain no occurrence of a continuation constant.

- $\Lambda_{(\kappa,\mu)} * \Pi_{(\kappa,\mu)}$ is the set of *processes* defined by the following grammar, modulo α-equivalence:

  Processes $p ::= t * \pi$ (where t is a closed λ_c-term and π is a stack)

- The *execution* $\prec_K$ which is the smallest preorder on the set of processes such that

$$\begin{array}{rcll} tu * \pi & \succ_K & t * u \bullet \pi & \text{(push)} \\ \lambda x.t * u \bullet \pi & \succ_K & t[x := u] * \pi & \text{(grab)} \\ cc * t \bullet \pi & \succ_K & t * k_\pi \bullet \pi & \text{(save)} \\ k_{\pi'} * t \bullet \pi & \succ_K & t * \pi' & \text{(restore)} \end{array}$$

Note that there is no evaluation rule for the special instructions, thus $\prec_K$ treats the special instructions as inert constants; depending on the contest we may define other evaluation relations with specific evaluation rules for the special instructions.

The cardinality of $\Lambda_{(\lambda,\mu)}$, $\Pi_{(\lambda,\mu)}$, $\mathcal{R}_{(\lambda,\mu)}$ and $\Lambda_{(\lambda,\mu)} * \Pi_{(\lambda,\mu)}$ is the maximum of the cardinality of λ, the cardinality of μ and $\aleph_0$.

A *realizability algebra* is a tuple $\mathcal{A} = (\kappa, \mu, \prec, \perp\!\!\!\perp)$, such that:

- κ and μ are ordinals (they fix the number of special instructions and stacks bottoms)
- $\prec$ is a preorder on the set of processes $\Lambda_{(\kappa,\mu)} * \Pi_{(\kappa,\mu)}$ that extends $\prec_K$
- $\perp\!\!\!\perp$ is a final segment of the set of processes, *i.e.* if $t * \pi \succ t' * \pi'$ and $t' * \pi' \in \perp\!\!\!\perp$, then $t * \pi \in \perp\!\!\!\perp$. It is called the *pole* of the realizability algebra.

We assume that $\mathcal{A}$ lives in a model $\mathcal{M}$ of ZFC. $\mathcal{M}$ is called the *ground model*.

2.2. **Realizability algebras and Forcing.** We mentioned in the introduction that classical realizability generalizes forcing, we briefly explain here how we can define a realizability algebra from a forcing boolean algebra – if the reader is not interested in the connection between classical realizability and forcing, they can skip this part and go directly to Subsection 2.3 –.

Roughly speaking, a forcing notion $\mathbb{B}$ corresponds to a special case of realizability algebra where the terms and the stacks bottoms correspond to the elements of $\mathbb{B}$; the constant cc corresponds to the maximal condition 1; the operations pq, $p \bullet q$ and $p * q$ all correspond to $p \wedge q$; each term k_p corresponds to p; the relation $p \succ q$ corresponds to $p \leq q$ (i.e. $p \wedge q = p$) and the only realizer is $\mathbf{1}$. More precisely, given a boolean algebra $\mathbb{B} = (B, 1, 0, \wedge, \vee, \neg)$ we can define a realizability algebra $\mathcal{A}_{\mathbb{B}} = (\kappa, \mu, \prec, \perp\!\!\!\perp)$ as

[3]In [3] [4] and [6], the set of realizers is denoted by $\mathcal{QP}$.

follows: $\kappa := 0$, so that there are no special instructions; μ is the cardinality of $\mathbb{B}$, so there is a stack bottom for every condition of $\mathbb{B}$ (it follows that for every $p \in \mathbb{B}$, we can represent p by the term $k_p \in \Lambda_{(0,\mu)}$). In order to define the preorder and the pole, we first define by induction a function $\tau : \Lambda^*_{(0,\mu)} \cup \Pi_{(0,\mu)} \to \mathbb{B}$ (where $\Lambda^*_{(0,\mu)}$ denotes the set of all – possibly open – λ_c-terms):

- for every stack bottom p, we let $\tau(p) := p$
- for every term t and every stack π, we let $\tau(t \bullet \pi) := \tau(t) \wedge \tau(\pi)$
- for every variable x, $\tau(x) := \tau(cc) := 1$
- for all λ_c-terms t, u we let $\tau(tu) := \tau(t) \wedge \tau(u)$
- for every variable x and every term t, we let $\tau(\lambda x.t) := \tau(t)$
- for every stack π, we let $\tau(k_\pi) := \tau(\pi)$

Then, we let $\prec$ be defined by $t_1 * \pi_1 \succ t_2 * \pi_2$ if and only if $\tau(t_1) \wedge \tau(\pi_1) \leq \tau(t_2) \wedge \tau(\pi_2)$ and we let $\perp\!\!\!\perp$ be the set of all processes $t * \pi$ such that $\tau(t) \wedge \tau(\pi) = 0$.

2.3. **Non extensional set theory.** In order to define a realizability model for set theory, we consider a non extensional version of set theory, the theory ZF_ε The language of ZF_ε is a first order language that has three binary relations ε, $\in$ and $\subseteq$. Formulas are built as usual from atomic formulas (including $\top, \bot$), with the only logical symbols $\Rightarrow, \forall$. We shall write $\neg F$ for $F \Rightarrow \bot$; $F \wedge G$ abbreviates $(F \Rightarrow G \Rightarrow \bot) \Rightarrow \bot$; $F \vee G$ abbreviates $(F \Rightarrow \bot) \Rightarrow (G \Rightarrow \bot) \Rightarrow \bot$. $F_1, \dots, F_n \Rightarrow F$ stands for $F_1 \Rightarrow (\cdots \Rightarrow (F_n \Rightarrow F) \cdots)$; $\exists x F$ abbreviates $\neg \forall x \neg F$; $\exists x\{F_1, \dots, F_n\}$ stands for $\neg \forall x (F_1, \dots, F_n \Rightarrow \bot)$. We shall write $a \not\varepsilon b$ for $a \,\varepsilon\, b \Rightarrow \bot$, and $a \notin b$ for $a \notin b \Rightarrow \bot$. The formula $\forall z (x \,\varepsilon\, z \Rightarrow y \,\varepsilon\, z)$ is written $x = y$; it is the *strong* or *Leibniz equality*. The formula $x \subseteq y \wedge y \subseteq x$ is written $x \simeq y$; it is the *weak* or *extensional equality*. The formula $\forall x (x \,\varepsilon\, a \Rightarrow F(x))$ is often written $\forall x \,\varepsilon\, a F(x)$, and the formula $\neg(\forall x F(x) \Rightarrow x \not\varepsilon a)$ is written $\exists x \,\varepsilon\, a F(x)$.

The axioms of ZF_ε are the following:

(0) Axioms of extensionality.

$$\forall x \forall y [x \in y \iff \exists z \;\varepsilon\; y (x \simeq z)];$$
$$\forall x \forall y [x \subseteq y \iff \forall z \;\varepsilon\; x (z \in y)].$$

(1) Axiom schema of foundation.

$$\forall x_1 \dots \forall x_n \forall a (\forall x (\forall y \;\varepsilon\; x (F[y, x_1, \dots, x_n] \Rightarrow F[x, x_1, \dots, x_n])) \Rightarrow F[a, x_1, \dots, x_n])$$

for every formula $F[x, x_1, \dots, x_n]$.

(2) Axiom schema of comprehension.

$$\forall a \exists b \forall x [x \;\varepsilon\; b \iff (x \;\varepsilon\; a \wedge F(x))]$$

(for every formula $F(x, x_1, \dots, x_n)$).

(3) Axiom of pairing.

$$\forall a \forall b \exists x [a \;\varepsilon\; x \wedge b \;\varepsilon\; x]$$

(4) Axiom of union.

$$\forall a \exists b \forall x \;\varepsilon\; a \forall y \;\varepsilon\; x (y \;\varepsilon\; b).$$

(5) Axiom schema of power set.

$$\forall a \exists b \forall x \exists y \;\varepsilon\; b \forall z (z \;\varepsilon\; y \iff (z \;\varepsilon\; a \wedge F(z, x)))$$

(for every formula $F(z, x, x_1, \dots, x_n)$).

(6) Axiom schema of collection.

$$\forall a \exists b \forall x \;\varepsilon\; a[\exists y F(x,y) \Rightarrow \exists y \;\varepsilon\; b\; F(x,y)]$$

(for every formula $F(x, y, x_1, \dots, x_n)$).

(7) Axiom schema of infinity.

$$\forall a \exists b \{a \;\varepsilon\; b \wedge \forall x \;\varepsilon\; b[\exists y F(x,y) \Rightarrow \exists y \;\varepsilon\; b\; F(x,y)]\}$$

(for every formula $F(x, y, x_1, \dots, x_n)$).

We say that a formula $F(x, \vec{w})$ is *extensional with respect to* x if ZF_ε proves $\forall \vec{w}\, \forall a, b\, (a \simeq b \Rightarrow F(a, \vec{w}) \Rightarrow F(b, \vec{w}))$.

It can be proven that ZF_ε is a conservative extension of ZF (see for instance [4]). Thus from any model $\mathcal{N}_\varepsilon = (|\mathcal{N}_\varepsilon|, \varepsilon, \in, \subseteq)$ of ZF_ε, by restricting the language and taking the quotient by $\simeq$, we get a model $\mathcal{N} = (|\mathcal{N}_\varepsilon|/\simeq, \in)$ of ZF in the usual sense. Moreover, any model of ZF can be obtained this way.

2.4. **The language of realizability.** We assume the consistency of the theory ZF enriched with the Axiom of Global Choice, thus we fix a model $\mathcal{M}$ of ZF with a global choice function g, i.e. g is a class function such that for every non-empty set x, we have $g(x) \in x$. The model $\mathcal{M}$ is called *the ground model.* We will define and study the realizability model within $\mathcal{M}$, for that we need to introduce the *language of realizability.* We fix in $\mathcal{M}$ a countably infinite set of first order variables. The set of *first order terms* is defined by induction as follows:

- every first order variable is a first order term
- $f(a_1, ..., a_n)$ is a first order term, for every $n \in \mathbb{N}$, every class function f from $\mathcal{M}^n$ to $\mathcal{M}$ and every n-tuple of first order terms $a_1, ..., a_n$.

The set $\mathcal{F}$ of formulas of the language of realizability is defined by induction as follows.

- If a, b are first order terms, then $a \neq b$, $a \not\varepsilon\, b$, $a \subseteq b$, $a \notin b$, $\top$, $\bot$ are in $\mathcal{F}$, and they are called the *atomic formulas.*[4]
- If A and B are in $\mathcal{F}$, and x is a first order variable, then $A \Rightarrow B$, $\forall x A$, $A \cap B$ and $A \cup B$ are in $\mathcal{F}$.[5]

As before, we shall write $\neg F$ for the formula $F \Rightarrow \bot$; $F \wedge G$ abbreviates $(F \Rightarrow G \Rightarrow \bot) \Rightarrow \bot$; $F \vee G$ abbreviates $(F \Rightarrow \bot) \Rightarrow (G \Rightarrow \bot) \Rightarrow \bot$. $F_1, \dots, F_n \Rightarrow F$ stands for $F_1 \Rightarrow (\cdots \Rightarrow (F_n \Rightarrow F) \cdots)$; $\exists x F$ abbreviates $\neg \forall x \neg F$; $\exists x\{F_1, \dots, F_n\}$ stands for $\neg \forall x (F_1, \dots, F_n \Rightarrow \bot)$. We shall write $a \;\varepsilon\; b$ for $a \not\varepsilon\, b \Rightarrow \bot$, and $a \in b$ for $a \notin b \Rightarrow \bot$. The formula $\forall z (x \not\varepsilon\, z \Rightarrow y \not\varepsilon\, z)$ is written $x = y$; the formula $x \subset y \wedge y \subset x$ is written $x \simeq y$. The formula $\forall x (\neg F(x) \Rightarrow x \not\varepsilon\, a)$ is written $\forall x \;\varepsilon\; a F(x)$, and the formula $\neg \forall x F(x) \Rightarrow x \not\varepsilon\, a$ is written $\exists x \;\varepsilon\; a F(x)$.

2.5. **Realizability models.** From now until the end of this section, we fix a realizability algebra $\mathcal{A} = (\lambda, \mu, \prec, \perp\!\!\!\perp)$ in $\mathcal{M}$. To simplify the notation we will write Λ and Π for $\Lambda_{(\lambda,\mu)}$ and $\Pi_{(\lambda,\mu)}$ respectively. For each closed formula φ in the language of realizability, we define a *truth value* $|\varphi| \subseteq \Lambda$ and a *falsity value* $\|\varphi\| \subseteq \Pi$ by induction on the length of the formula.

[4]For technical reasons that will be much clear in the following, it is better to take as atomic formulas the negative expressions $a \neq b$, $a \not\varepsilon\, b$, $a \notin b$, rather than $a = b$, $a \;\varepsilon\; b$ and $a \in b$.

[5]Roughly, the connectives $\cap$ and $\cup$ correspond to special conjunctions and disjunctions.

Definition 2.1.
- $t \in |\varphi| \iff \forall \pi \in \|\varphi\| (t * \pi \in \perp\!\!\!\perp)$
- $\|\top\| = \emptyset$, $\|\bot\| = \Pi$, $\|a \not\varepsilon b\| = \{\pi \in \Pi;\ (a, \pi) \in b\}$
- $\|a \subseteq b\|$ and $\|a \notin b\|$ are defined simultaneously by induction on $(\text{rk}(a) \cup \text{rk}(b), \text{rk}(a) \cap \text{rk}(b))$ ($\text{rk}(a)$ being the rank of a):
  - $\|a \subseteq b\| = \{t \bullet \pi;\ (t, \pi) \in \Lambda \times \Pi,\ (c, \pi) \in a \text{ and } t \in |c \notin b|\}$
  - $\|a \notin b\| = \{t \bullet t' \bullet \pi;\ (t, t', \pi) \in \Lambda \times \Lambda \times \Pi,\ (c, \pi) \in b,\ t \in |a \subseteq c|,\ t' \in |c \subseteq a|\}$
- $\|A \Rightarrow B\| = \{t \bullet \pi; (t, \pi) \in \Lambda \times \Pi,\ t \in |A|, \pi \in \|B\|\}$
- $\|\forall x A\| = \{\pi \in \Pi;\ \exists a (\pi \in \|A[a/x]\|)\}$
- $\|A \cap B\| = \|A\| \cup \|B\|$
- $\|A \cup B\| = \|A\| \cap \|B\|$

We write $t \Vdash \varphi$ for $t \in |\varphi|$.

We can think at $\|\varphi\|$ as the set of all the stacks who 'witnesses' the falsity of the formula; since no stack can witness that $\top$ is false and every stack can witness that $\bot$ is false, we have $\|\top\| = \emptyset$ and $\|\bot\| = \Pi$. The falsity value of $a \not\varepsilon b$ contains all the stacks π such that $(a, \pi) \in b$, namely all the stacks that '*witness*' that a belongs to the transitive closure of b;[6] the definition of the falsity value for the other formulas can be justified by similar arguments by considering the axioms of ZF_ε. On the other hand, a realizer of a certain formula is a term which is somehow 'incompatible' with every stack which witnesses the falsity of the formula, namely the realizer and the stack form a process which belongs to the pole.

We say that a formula φ is *realized* if there is $t \in \mathcal{R}$ such that $t \Vdash \varphi$. We often omit the realizers and write $\Vdash \varphi$ to say that φ is realized (by some realizer).

The following theorems establish that the axioms of ZF_ε are realized and that the set of closed formulas that are realized forms a consistent theory (see Krivine [4] for a detailed proof).

Theorem 2.2. *(Krivine [4]) (Adequacy lemma) Let $A_1, ..., A_n, A$ be closed formulas of ZF_ε and suppose that $x_1 : A_1, ..., x_n : A_n \vdash t : A$ (with the rules of natural deduction). If $\xi_1 \Vdash A_1, ..., \xi_n \Vdash A_n$, then $t[\xi_1 := x_1, ..., \xi_n := x_n] \Vdash A$. In particular, if $\vdash t : A$, then $t \Vdash A$.*

Theorem 2.3. *(Krivine [4]) Peirce law is realized, namely for every formulas A, B, we have $cc \Vdash ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$.*

Theorem 2.4. *(Krivine [4]) The axioms of ZF_ε are realized.*

The realizability algebra $\mathcal{A}$ is said to be *consistent* if for every realizer t, there exists a stack π such that $t \star \pi \notin \perp\!\!\!\perp$. This is equivalent to saying that the formula $\bot$ is not realized.

The *realizability theory of* $\mathcal{A}$ is the set of all closed formulas that are realized. Models of this theory are called *pre-realizability models of* $\mathcal{A}$; by Theorem 2.4, they are models of ZF_ε. Theorems 2.2 and 2.3 imply that this theory is closed under classical deduction. Therefore, there exists a pre-realizability model of $\mathcal{A}$ if and only if $\mathcal{A}$ is consistent. Given any pre-realizability model $\mathcal{N}_\varepsilon$, the corresponding model of ZF is called the *realizability model* associated with $\mathcal{N}_\varepsilon$.

[6]Now, it should be clear why we chose to take $a \not\varepsilon b$ and $a \notin b$ as atomic formulas, in fact it is easier to define what it a 'witness' of the falsity of $a \not\varepsilon b$ or $a \notin b$ rather than a 'witness' of the falsity of $a \varepsilon b$ or $a \in b$.

Notation 2.5. Given two first order terms a, b and A a formula of the realizability language, we write $a = b \hookrightarrow A$ for the formula $(a \neq b) \cup A$.

It follows from Definition 2.1, that

$$\|a = b \hookrightarrow A\| := \begin{cases} \emptyset & \text{if } a \neq b \\ \|F\| & \text{otherwise} \end{cases}$$

It is proved in [4] that $a = b \hookrightarrow A$ can be realized to be equivalent to $a = b \Rightarrow A$; yet $a = b \hookrightarrow A$ is simpler to realize.

2.6. **Special function symbols.** We defined the language of realizability in such a way that every function or class function definable in $\mathcal{M}$ is a first order term of the language. In particular, we will make use of the following class functions.

- pair denotes the class function defined by $\text{pair}(a, b) := (a, b)$ for every $a, b \in \mathcal{M}$.
- img is defined by $\text{img}(f) := \{(y, \pi);\ \exists x\ (\text{pair}(x, y), \pi) \in f\}$ for every $f \in \mathcal{M}$. The, the formula $\forall f \forall y\ (y \varepsilon \text{img}(f) \iff \exists x\ \text{pair}(x, y) \varepsilon f)$ is realized.

A useful property of forcing is that every set in a forcing model has a "name", that is there is a (first order) term in the language of forcing that is interpreted as this set in the forcing model. On the contrary, the elements of a realizability model do not necessarily have a "name", namely in general, for a given set a in a realizability model there is no first order term of the language of realizability that is interpreted as a. Nevertheless, for every definable element x of the realizability model, the singleton $\{x\}$ has a name, as stated in the following lemma.

Lemma 2.6. *(Naming of singletons) For every formula $A(w_1, ..., w_n, x)$ in the language of realizability, we can define a class function s_A from $\mathcal{M}^n$ to $\mathcal{M}$ such that the following formulas are realized:*

(1) $\forall \vec{a} \forall x (x \varepsilon s_A(\vec{a}) \Rightarrow A(\vec{a}, x))$
(2) $\forall \vec{a} (\exists x A(\vec{a}, x) \Rightarrow \exists x (x \varepsilon s_A(\vec{a})))$

In particular, when $A(\vec{a}, x)$ defines a singleton, $s_A(\vec{a})$ denotes that singleton.

Proof. We work in $\mathcal{M}$. By using the global choice function g, we can define a (class) function $f : \mathcal{M}^n \times \Lambda \to \mathcal{M}$ such that for every $\vec{a}$ in $\mathcal{M}^n$ and every term $t \in \Lambda$, the set $f(\vec{a}, t)$ is an element in the class $\{x \in \mathcal{M};\ t \Vdash A(\vec{a}, x)\}$ unless this class is empty (in which case $f(\vec{a}, t)$ is undefined). We let

$$s_A(\vec{a}) := \{(f(\vec{a}, t), t \bullet \pi);\ \pi \in \Pi, t \in \Lambda \text{ and } f(\vec{a}, t) \text{ is defined}\}.$$

Then, for every set $b \in \mathcal{M}$, we have $\|\neg A(\vec{a}, b)\| \supseteq \|b \not\varepsilon s_A(\vec{a})\|$. It follows that the identity realizes $\forall \vec{w} \forall x ((A(\vec{w}, x) \Rightarrow \bot) \Rightarrow x \not\varepsilon s_A(\vec{w}))$, that proves the first claim. Moreover, $\|\forall x (x \not\varepsilon s_A(\vec{a}))\| = \|\forall x (A(\vec{a}, x) \Rightarrow \bot)\|$, thus the identity realizes $\forall \vec{w} \forall x (x \not\varepsilon s_A(\vec{w}) \Rightarrow (A(\vec{w}, x) \Rightarrow \bot))$, that proves the second claim. □

3. Representing the ordinals in realizability models

From now on, we fix a model $\mathcal{M}$ of ZFC, which will be our ground model, and an infinite ordinal κ in $\mathcal{M}$.

In this section, we will define a realizability model in which every ordinal up to κ has a representative; this is crucial for proving our main result.

In order to simplify the notation, we write Λ and Π for $\Lambda_{(\kappa,1)}$ and $\Pi_{(\kappa,1)}$ respectively. We fix an enumeration $(\nu_\alpha)_{\alpha<\kappa}$ of the terms and we let χ be a special instruction (*e.g.* the first special instruction ξ_0)

Definition 3.1. We define $\succ$ as the smallest pre-order on $\Lambda * \Pi$ which extends $\succ_K$ and such that for every pair of ordinals α, β, every terms t, u, v and every stack π:

(1) if $\alpha < \beta$, then $\chi * \nu_\alpha \bullet \nu_\beta \bullet t \bullet u \bullet v \bullet \pi \succ t \bullet \pi$
(2) if $\alpha = \beta$, then $\chi * \nu_\alpha \bullet \nu_\beta \bullet t \bullet u \bullet v \bullet \pi \succ u \bullet \pi$
(3) if $\alpha > \beta$, then $\chi * \nu_\alpha \bullet \nu_\beta \bullet t \bullet u \bullet v \bullet \pi \succ v \bullet \pi$

Let $\perp\!\!\!\perp$ be any final segment of the set of processes. We define $\mathcal{A}_\kappa^{\perp\!\!\!\perp}$ as the realizability algebra $(\kappa, 1, \prec, \perp\!\!\!\perp)$. Let $\mathcal{N}_\varepsilon(\kappa)$ be any pre-realizability model[7] of $\mathcal{A}_\kappa^{\perp\!\!\!\perp}$ and let $\mathcal{N}(\kappa)$ be the corresponding realizability model. For every ordinal $\alpha \leq \kappa$, we define $\widehat{\alpha} := \{(\widehat{\beta}, \nu_\beta \bullet \pi);\ \pi \in \Pi, \beta < \alpha\}$. We are going to show that for every ordinal $\alpha \leq \kappa$ in the ground model, $\widehat{\alpha}$ names an ordinal in the realizability model and if $\alpha < \beta$ are two ordinals in the ground model with $\beta \leq \kappa$, then $\widehat{\alpha}\ \varepsilon\ \widehat{\beta}$ is realized; in this sense $\hat{\alpha}$ is a representative of α.

Lemma 3.2. *Let $\beta < \alpha \leq \kappa$, then $\lambda x.x(\nu_\beta) \Vdash \widehat{\beta}\ \varepsilon\ \widehat{\alpha}$*

Proof. Let $t \in |\widehat{\beta}\ \varepsilon\ \widehat{\alpha}|$ and $\pi \in \Pi$. By definition, $\nu_\beta \bullet \pi \in \|\widehat{\beta}\ \varepsilon\ \widehat{\alpha}\|$, therefore $t * \nu_\beta \bullet \pi \in \perp\!\!\!\perp$ and $\lambda x.x(\nu_\beta) * t \bullet \pi \in \perp\!\!\!\perp$. □

Notation. For every ordinal $\alpha \leq \kappa$, every formula A of the language of realizability and every first order term b, we write $b\ \varepsilon\ \widehat{\alpha} \hookrightarrow A$ for the formula $(b \not\varepsilon\ \widehat{\alpha}) \cup (\top \Rightarrow A)$.

Remark 3.3. Let $\alpha \leq \kappa$, let b be a closed first order term and A a closed formula of the language of realizability. If b is of the form $\hat{\beta}$ for an ordinal $\beta < \alpha$, then $\|b\ \varepsilon\ \hat{\alpha} \hookrightarrow A\| = \{\nu_\beta \bullet \pi;\ \pi \in \|A\|\}$, else $\|b\ \varepsilon\ \hat{\alpha} \hookrightarrow A\| = \emptyset$.

Lemma 3.4. *For every $\alpha \leq \lambda$, every first order term $b(x_1, \ldots, x_n)$ and every formula $F(x_1, \ldots, x_n)$ in the language of realizability,*

$$\Vdash \forall x_1, \ldots, x_n\ \left((b\ \varepsilon\ \widehat{\alpha} \hookrightarrow F) \iff (b\ \varepsilon\ \widehat{\alpha} \Rightarrow F)\right).$$

Proof. To simplify the notation, we can assume that F and b are closed. Let $d := \lambda t.\lambda u.t(\lambda x.xu)$, we show that $d \Vdash (b\ \varepsilon\ \widehat{\alpha} \Rightarrow F) \Rightarrow (b\ \varepsilon\ \widehat{\alpha} \hookrightarrow F)$. If b is not of the form $\hat{\beta}$ for some ordinal $\beta < \alpha$, then $\|b\ \varepsilon\ \hat{\alpha} \hookrightarrow F\| = \|\top\|$, thus the implication above is realized. Suppose then that b is of the form $\hat{\beta}$ for some ordinal $\beta < \alpha$. Then by the remark above, we have $\|b\ \varepsilon\ \hat{\alpha} \hookrightarrow F\| = \{\nu_\beta \bullet \pi;\ \pi \in \|F\|\}$. Let $\xi \in \Lambda$ such that $\xi \Vdash b\ \varepsilon\ \hat{\alpha} \Rightarrow F$ and let $\nu_\beta \bullet \pi \in \|b\ \varepsilon\ \hat{\alpha} \hookrightarrow F\|$, then by Lemma 3.2 we have $\lambda x.x(\nu_\beta) \Vdash \hat{\beta}\ \varepsilon\ \hat{\alpha}$. It follows that $\xi * \lambda x.x(\nu_\beta) \bullet \pi \in \perp\!\!\!\perp$. We have $d * \xi \bullet \nu_\beta \bullet \pi \succ \xi(\lambda x.x(\nu_\beta)) * \pi \succ \xi * (\lambda x.x(\nu_\beta)) \bullet \pi \in \perp\!\!\!\perp$, this completes the proof that $d \Vdash (b\ \varepsilon\ \widehat{\alpha} \Rightarrow F) \Rightarrow (b\ \varepsilon\ \widehat{\alpha} \hookrightarrow F)$.

Let $g := \lambda t.\lambda u.\, \text{cc}(\lambda k.u(\lambda x.k(tx)))$, we show that $g \Vdash (b\ \varepsilon\ \widehat{\alpha} \hookrightarrow F) \Rightarrow (b\ \varepsilon\ \widehat{\alpha} \Rightarrow F)$. Let $t, u \in \Lambda$ and $\pi \in \Pi$ such that $t \Vdash b\ \varepsilon\ \widehat{\alpha} \hookrightarrow F$, $u \Vdash b\ \varepsilon\ \widehat{\alpha}$ and $\pi \in \|F\|$. We have $g * t \bullet u \bullet \pi \succ u * (\lambda x.k_\pi(tx)) \bullet \pi$ while $u \Vdash b \not\varepsilon\ \widehat{\alpha} \Rightarrow \perp\!\!\!\perp$, so it is enough to prove that $\lambda x.k_\pi(tx)$ realises $b \not\varepsilon\ \widehat{\alpha}$. If $b = \widehat{\beta}$ for some $\beta < \alpha$, then for every stack π, we have $\lambda x.k_\pi(tx) * \nu_\beta \bullet \pi' \succ t * \nu_\beta \bullet \pi \in \perp\!\!\!\perp$. Otherwise, we have $\|b \not\varepsilon\ \widehat{\alpha}\| = \emptyset$, which completes the proof. □

[7] We're trying to prove something about *all* realizability models, so it makes sense not to assume right away that $\mathcal{A}_\kappa^{\perp\!\!\!\perp}$ is consistent: see the remark at the end of this section.

Notation. For every $\alpha \leq \lambda$ and every formula F, we write $\forall x^{\widehat{\alpha}} F$ for the formula $\forall x (x \mathrel{\varepsilon} \widehat{\alpha} \hookrightarrow F)$ and we write $\exists x^{\widehat{\alpha}} F$ for the formula $\forall x (x \mathrel{\varepsilon} \widehat{\alpha} \hookrightarrow F \Rightarrow \bot) \Rightarrow \bot$

Notation. We write $\mathrm{Ord}_\in(a)$ for the following formula, which says that a is a transitive set (with respect to $\in$) and that $\in$ is a strict linear order relation on a:

$$\begin{array}{lll} & \forall x{\in}a\ \forall y{\in}x\ y{\in}a & (a \text{ is a transitive set}) \\ \wedge & \forall x{\in}a\ \forall y{\in}a\ \forall z{\in}a\ (x{\in}y \Rightarrow y{\in}z \Rightarrow x{\in}z) & (\in \text{ defines a strict total order on } a) \\ \wedge & \forall x{\in}a\ \forall y{\in}a\ (x{\in}y \vee x{\simeq}y \vee y{\in}x) & (\text{this order is linear}). \end{array}$$

This amounts to saying that a is an ordinal in the usual sense, since the axiom of foundation guarantees that $\in$ is well-founded.

We can also define a non-extensional version of this formula as follows.

Notation. We write $\mathrm{Ord}_\varepsilon(a)$ for the following formula, which says that a is a transitive set (with respect to ε) and that ε is a strict linear order relation on a:

$$\begin{array}{lll} & \forall x \varepsilon a\ \forall y \varepsilon x\ y \varepsilon a & (a \text{ is a transitive set}) \\ \wedge & \forall x \varepsilon a\ \forall y \varepsilon a\ \forall z \varepsilon a\ (x \varepsilon y \Rightarrow y \varepsilon z \Rightarrow x \varepsilon z) & (\varepsilon \text{ defines a strict total order on } a) \\ \wedge & \forall x \varepsilon a\ \forall y \varepsilon a\ (x \varepsilon y \vee x{=}y \vee y \varepsilon x) & (\text{this order is linear}). \end{array}$$

It is easier to work with the non-extensional version because the definition of its falsity value is simpler). As it turns out, the non-extensional version implies the extensional one:

Proposition 3.5. $\mathrm{ZF}_\varepsilon \vdash \forall a\ (\mathrm{Ord}_\varepsilon(a) \Rightarrow \mathrm{Ord}_\in(a))$.

Proof. Let us fix a model of ZF_ε. Let a be a non-extensional ordinal (*i.e.* let a be such that $\mathrm{Ord}_\varepsilon(a)$ is true).

Let $x \in a$ and $y \in x$. Let $x' \simeq x$ such that $x' \mathrel{\varepsilon} a$. Then $y \in x'$, so let $y' \simeq y$ such that $y' \mathrel{\varepsilon} x'$. Then $y' \mathrel{\varepsilon} x'$ and $x' \mathrel{\varepsilon} a$, so $y' \mathrel{\varepsilon} a$. Since $y' \simeq y$, we have $y \in a$.

Let $x, y, z \in a$ such that $x \in y$ and $y \in z$. Let $z' \simeq z$ such that $z' \mathrel{\varepsilon} a$. We have $y \in z'$, so let $y' \simeq y$ such that $y' \mathrel{\varepsilon} z'$. We have $x \in y'$, so let $x' \simeq x$ such that $x' \mathrel{\varepsilon} y'$. Then $y' \mathrel{\varepsilon} z'$ and $z' \mathrel{\varepsilon} a$, so $y' \mathrel{\varepsilon} a$. Moreover, $x' \mathrel{\varepsilon} y'$ and $y' \mathrel{\varepsilon} a$, so $x' \mathrel{\varepsilon} a$. Therefore, $x' \in z'$, with $x' \simeq x$ and $z' \simeq z$, which means that $x \in z$.

Let $x, y \in a$. Let $x' \simeq x$ and $y' \simeq y$ such that $x' \mathrel{\varepsilon} a$ and $y' \mathrel{\varepsilon} a$. If $x' \mathrel{\varepsilon} y'$ then $x \in y$, if $x' = y'$ then $x \simeq y$, and if $y' \mathrel{\varepsilon} x'$ then $y \in x$. □

Non-extensional ordinals have a very useful property: they ε-contain a $=$-unique representative of each of their $\in$-elements:

Proposition 3.6. $\mathrm{ZF}_\varepsilon \vdash \forall a\ (\mathrm{Ord}_\varepsilon(a) \Rightarrow \forall x \mathrel{\varepsilon} a\ \forall y \mathrel{\varepsilon} a\ (x \simeq y \Rightarrow x = y))$.

Proof. Let a be a non-extensional ordinal. Let x and y be ε-elements of a such that $x \neq y$. Then one must have $x \mathrel{\varepsilon} y$ or $y \mathrel{\varepsilon} x$. In the fist case, we get $x \in y$, and in the second, $y \in x$. Because of the extensional axiom of well-foundation, in either case, one must have $x \not\simeq y$ □

Proposition 3.7. *For every $\alpha \leq \lambda$, the formula $\mathrm{Ord}_\varepsilon(\widehat{\alpha})$ is realized.*

Proof. First we show that $\widehat{\alpha}$ is transitive, namely we show that $\forall x^{\widehat{\alpha}} \forall y (y \notin \widehat{\alpha} \Rightarrow y \notin x)$ is realised by the term $\theta = \lambda t.\lambda u.u$. Let $\beta < \alpha$, $c \in M$, $u \in |y \notin \widehat{\alpha}|$ and $\pi \in \|c \notin \widehat{\beta}\|$, we want to show that $\theta * \nu_\beta \centerdot u \centerdot \pi \in \perp\!\!\!\perp$. We have $\delta * \nu_\beta \centerdot u \centerdot \pi \succ u * \pi$. Moreover, $\pi \in \|c \notin \widehat{\beta}\|$, so $\|c \notin \widehat{\beta}\|$ is not empty. Thus there exists $\gamma < \beta < \alpha$ such that $c = \widehat{\gamma}$, therefore $\|c \notin \widehat{\beta}\| = \{\nu_\gamma \centerdot \pi';\ \pi' \in \Pi\} = \|c \notin \widehat{\alpha}\|$, hence $u * \pi \in \perp\!\!\!\perp$.

An analogous argument shows that ε defines a strict order on $\widehat{\alpha}$. We show that this order is total. For that, we are going to use the instruction χ. We let

$$\tau := \lambda b.\lambda c.\lambda t.\lambda u.\lambda v.\chi bc(tb)(u)(vc)$$

and we show that $\tau \Vdash \forall x^{\widehat{\alpha}} \forall y \widehat{\alpha}(x \notin y \Rightarrow x \neq y \Rightarrow y \notin x \Rightarrow \bot)$. Let $\beta, \gamma < \alpha$, $t \in |\widehat{\beta} \notin \widehat{\alpha}|$, $u \in |\widehat{\beta} \neq \widehat{\alpha}|$, $v \in |\widehat{\alpha} \notin \widehat{\beta}|$ and $\pi \in \Pi$. If $\beta < \gamma$, then $\tau * \nu_\beta \bullet \nu_\gamma \bullet t \bullet u \bullet v \bullet \pi \succ t * \nu_\beta \bullet \pi \in \perp\!\!\!\perp$, because $t \Vdash (\widehat{\beta} \notin \gamma) \wedge (\beta < \gamma)$. Thus $\tau * \nu_\beta \bullet \nu_\gamma \bullet t \bullet u \bullet v \bullet \pi \in \perp\!\!\!\perp$. The proof is analogous if $\gamma < \beta$. Suppose $\beta = \gamma$, then $\tau * \nu_\beta \bullet \nu_\gamma \bullet t \bullet u \bullet v \bullet \pi \succ v * \pi \in \perp\!\!\!\perp$, because $v \Vdash (\widehat{\beta} \neq \widehat{\gamma})$ implies $v \Vdash \bot$. □

As an immediate consequence of Proposition 3.6 and Proposition 3.5, we get the following result.

Corollary 3.8. *For every $\alpha \leq \lambda$, the formula* $\mathrm{Ord}_\in(\widehat{\alpha})$ *is realized.*

Remark 3.9. It can be proved that there exists a final segment $\perp\!\!\!\perp$ such that $\mathcal{A}_\kappa^{\perp\!\!\!\perp}$ is consistent (for example, take $\perp\!\!\!\perp = \emptyset$). Moreover, it can be proved [1] that there exists a final segment $\perp\!\!\!\perp$ such that $\mathcal{A}_\kappa^{\perp\!\!\!\perp}$ has a pre-realizability model which satisfies "$\daleth 2$ has more than 2 elements".

4. Zorn's lemma restricted to a single ordinal

4.1. **Zorn's lemma and the Axiom of Choice.** Given any ordinal κ, we call *Zorn's lemma restricted to* κ the following statement: "let X be a non-empty set and R a strict partial order on X such that every R-chain (i.e. every linearly ordered subset of X) has a strict upper bound. Then there is an R-increasing κ-sequence (i.e. a sequence $(s_\beta)_{\beta<\kappa}$ of elements of X such that for all $\beta < \alpha < \kappa$, $s_\beta \; R \; s_\alpha$)". For every ordinal κ in the ground model, we are going to prove that *Zorn's lemma restricted to* $\widehat{\kappa}$ is true in the realizability model $\mathcal{N}_\kappa$ defined in the previous section.

We shall first show that AC is equivalent to "for every ordinal κ, ZL_κ".

Proposition 4.1. *Zorn's lemma is true if and only if for all κ, Zorn's lemma restricted to κ is true.*

Proof. Assume that for all κ, Zorn's lemma restricted to κ is true. Let X be a set and R a strict partial order on X such that every R-chain has a non-necessarily strict upper bound. Let κ be the least ordinal such that there is no injective function from κ to X. Then, there cannot be an R-increasing κ-chain so, by ZL_κ, there must be an R-chain C with an upper bound y but no strict upper bound: this means that y is maximal.

Now, assume Zorn's lemma. Let X be a non-empty set and R a strict partial order on X. By using Zorn's lemma, we show that the statement "every R-chain has a strict upper bound" is false, thus for every ordinal κ, the premises of ZL_κ is false, hence ZL_κ is true. Suppose by contradiction that every R-chain has a strict upper bound, then by Zorn's Lemma there exists a maximal element c of X. But $\{c\}$ is an R-chain and the maximality of c implies that $\{c\}$ has no strict upper bound, a contradiction. □

Zorn's lemma restricted to a single ordinal κ is in fact equivalent to the principle of κ dependent choice, denoted DC_κ. We recall that the axiom DC_κ is the following statement: "let X be a non-empty set and R a binary relation such that for every $\alpha < \kappa$ and every α-sequence $s = (s_\beta)_{\beta<\alpha}$ of elements of X, there exists $y \in X$ such that $s \; R \; y$, then there is a κ-sequence $(s_\beta)_{\beta<\kappa}$ of elements of X such that for all $\alpha < \kappa$, $(s_\beta)_{\beta<\alpha} \; R \; s_\alpha$".

Proposition 4.2. *Let κ be an ordinal. Zorn's lemma restricted to κ is equivalent to the principle of κ dependent choices.*

Proof. First, assume Zorn's lemma restricted to κ. Let X be a non-empty set and R a binary relation such that for every $\alpha < \kappa$ and every α-sequence $s = (s_\beta)_{\beta<\alpha}$ of elements of X, there exists $y \in X$ such that $s\ R\ y$. Given any $\alpha \leq \kappa$, we will say that an α-sequence $(s_\beta)_{\beta<\alpha}$ of elements of X is *compatible with* R if for all $\beta < \alpha$, $(s_\gamma)_{\gamma<\beta}\ R\ s_\beta$. We assume by contradiction that there is no κ-sequence compatible with R..

Let Y be the set of all α-sequences compatible with R for all $\alpha < \kappa$. Let S be the strict prefix ordering on Y. We show that any S-chain has a strict upper bound. Let C be any S-chain and let $s = (s_\beta)_{\beta<\alpha}$ be its union: it is an α-chain compatible with R, for some $\alpha \leq \kappa$. We assumed that there is no κ-sequence compatible with R, so we must have $\alpha < \kappa$ and therefore $s \in Y$. Let s_α be such that $s\ R\ s_\alpha$: then $(s_\beta)_{\beta\leq\alpha}$ is a strict upper bound of C.

By ZL_κ applied to (Y, S), there is a κ-sequence of elements of Y compatible with S, hence a κ-sequence of elements of X, that leads us to a contradiction.

Now, assume the principle of κ dependent choices. Let X be a non-empty set and R a strict partial order on X such that every R-chain has a strict upper bound. Let S be the following relation: given any sequence $s = (s_\beta)_{\beta<\alpha}$ of elements of X with $\alpha < \kappa$ and any element y of X, $s\ S\ y$ if and only if:

- for all $\beta < \alpha$, $s_\beta\ R\ y$,
- *or* there exist $\gamma < \beta < \alpha$ such that $s_\gamma \not R\ s_\beta$.

The second condition is necessarily true whenever $\{\ s_\beta;\ \beta < \alpha\ \}$ is not a chain. Therefore, for every $\alpha < \kappa$ and every α-sequence $s = (s_\beta)_{\beta<\alpha}$ of elements of X, there does exist an $y \in X$ such that $s\ S\ y$.

By DC_κ applied to (X, S), there is a κ-sequence $s = (s_\alpha)_{\alpha<\kappa}$ compatible with S. By induction, one can prove that for all $\beta < \alpha < \kappa$, $s_\beta\ R\ s_\alpha$. □

The axiom of choice is equivalent to "for every ordinal κ, DC_κ" (see for instance [2]). It follows, once again, that AC is equivalent to "for every ordinal κ, ZL_κ".

Now, we will formalize these restricted Zorn's lemmas within ZF_ε.

Notation 4.3. (1) For every formula $R(\vec{w}, x, y)$ of ZF_ε we denote by $\mathrm{Upper}^R(\vec{w}, c, m)$ the formula $\forall x{\in}c\ R(\vec{w}, c, m)$.
(2) For every formula $R(\vec{w}, x, y)$ of ZF_ε, we denote by $\mathrm{Chain}^R(\vec{w}, c)$ the formula $\forall x{\in}c\ \forall y{\in}c\ (x \neq y \Rightarrow R(\vec{w}, x, y) \vee R(\vec{w}, y, x))$

Definition 4.4. Let α be a first-order term. A *Zorn schema restricted to* α is a set of formulas of ZF_ε which contains $\mathrm{Ord}(\alpha)$ and such that for every formula $R(\vec{w}, x, y)$ of ZF_ε which is extensional with respect to x and y, there exists a formula $Z_R(\vec{w}, \beta, \gamma)$ which is extensional with respect to β and y and such that the set contains the following formulas:

(1) $\forall\vec{w}\ (\forall c\ (\mathrm{Chain}^R(\vec{w}, c) \Rightarrow \exists m\ \ \mathrm{Upper}^R(\vec{w}, c, m)) \Rightarrow \forall\beta{\in}\alpha\ \exists y\ Z_R(\vec{w}, \beta, y))$,
(2) $\forall\vec{w}\ \forall\beta{\in}\alpha\ \forall\gamma{\in}\alpha\ \forall y\ \forall z\ (Z_R(\vec{w}, \beta, y) \Rightarrow Z_R(\vec{w}, \gamma, z) \Rightarrow \beta \in \gamma \Rightarrow R(\vec{w}, y, z))$
(3) $\forall\vec{w}\ \forall\beta{\in}\alpha\ \forall\gamma{\in}\alpha\ \forall y\ \forall z\ (Z_R(\vec{w}, \beta, y) \Rightarrow Z_R(\vec{w}, \gamma, z) \Rightarrow \beta \simeq \gamma \Rightarrow y \simeq z)$

Remark 4.5. Any Zorn schema restricted to α implies Zorn's lemma restricted to α. Indeed, let $\mathrm{Pair}(p, x, y)$ be a formula of ZF_ε which states that p is (extensionally equivalent to) the Kuratowski encoding of the ordered pair (x, y). Then we simply have to define $R(r, x, y)$ as the formula $\exists p\ (\mathrm{Pair}(p, x, y) \wedge p \in r)$.

4.2. **A non-extensional version of the Axiom of Choice.** In this subsection, we discuss a non-extensional version of the Axiom of Choice, called NEAC. We take $\mathcal{N}_\varepsilon(\kappa)$ as in the previous section, and we will we show that it satisfies NEAC.

We write Func(f) for the formula $\forall x, y, y'((x, y) \ \varepsilon \ f, (x, y') \ \varepsilon \ f \Rightarrow y = y')$. In other words, Func$(f)$ means that f is a functional in the sense of the strong equality $=$. On the other hand, Func(f) does not imply that f is compatible with the weak equality $\simeq$, namely if $(x, y) \in f$ and $(x', y') \in f$, then $x \simeq x'$ does not imply $y \simeq y$.

NEAC is the following statement:

$$\forall z \ \exists f \ ((\forall w \varepsilon f \ (w \varepsilon z)) \wedge \mathrm{Func}(f) \wedge \forall x, y \ \exists y'((x, y) \ \varepsilon \ z \Rightarrow (x, y') \ \varepsilon \ f))$$

In other words, NEAC establishes that every binary relation can be refined into a function in the non-extensional sense. It should be clear that the extensional analog of NEAC is the Axiom of Choice: suppose that $\{A_i\}_{i \in I}$ is a family of non empty sets, let R be the binary relation defined by $(x, A_i) \in R$ if and only if $x \in A_i$; if R can be refined into a function f compatible with the extensional equality, then f is a choice function. Neverthless, NEAC is not equivalent to the Axiom of Choice (considered as a formula of ZF_ε) as there are many examples of pre-realizability models where NEAC holds but AC fails (see [3], [4], [6]). We want to show that NEAC holds in the pre-realizability model $\mathcal{N}_\varepsilon(\kappa)$.

Proposition 4.6. *For every formula $A(w_1, ..., w_n, x)$ there is a functional relation f_A definable in $\mathcal{M}$ and of arity $n + 1$ such that the following formula is realized:* $\forall \vec{w}(\exists x \ A(\vec{w}, x) \Rightarrow \exists a \ \varepsilon \ \widehat{\kappa} \ A(\vec{w}, f_A(\vec{w}, a)))$

Proof. For every term t, we let $P_t := \{\pi; \ t * t \bullet \pi \notin \perp\!\!\!\perp\}$. For every $\vec{w} \in M$ and every $\alpha < \kappa$ such that $P_{\nu_\alpha} \cap \|\forall x \neg A(\vec{w}, x)\| \neq \emptyset$, we fix $x \in M$ such that $P_{\nu_\alpha} \cap \|\neg A(\vec{w}, x)\| \neq \emptyset$ and we let $f_A(\vec{w}, \widehat{\alpha}) = x$. Then, we extend f_A arbitrarily on M^{n+1}.

We show that $\lambda y.yy \Vdash \forall \vec{w}(\forall a^{\widehat{\kappa}} \neg A(\vec{w}, f_A(\vec{w}, a)) \Rightarrow \forall x \neg A(\vec{w}, x))$. Let $\vec{w}$ be a tuple of sets in $\mathcal{M}$, and let $t \in |\forall a^{\widehat{\kappa}} \neg A(\vec{w}, f_A(\vec{w}, a))|$ and $\pi \in \|\forall x \neg A(\vec{w}, x)\|$, we want to show that $t * t \bullet \pi \in \perp\!\!\!\perp$.

Let β be such that $t = \nu_\beta$. Since $t \Vdash \forall a^{\widehat{\kappa}} \neg A(\vec{w}, f_A(\vec{w}, a))$, in particular for every $\pi' \in \|\neg A(\vec{w}, f_A(\vec{w}, \widehat{\beta}))\|$, we have $t * t \bullet \pi' \in \perp\!\!\!\perp$. This means that $P_t \cap \|\neg A(\vec{w}, f_A(\vec{w}, \widehat{\beta}))\| = \emptyset$ thus $P_t \cap \|\forall x \neg A(\vec{w}, x)\| = \emptyset$. Since $\pi \in \|\forall x \neg A(\vec{w}, x)\|$, we have $\pi \notin P_t$, thus $t * t \bullet \pi \in \perp\!\!\!\perp$. □

From this proposition we can show that NEAC holds in $\mathcal{N}_\varepsilon(\kappa)$.

Corollary 4.7. *NEAC holds in $\mathcal{N}_\varepsilon(\kappa)$.*

Proof. Given a set a in $\mathcal{N}_\varepsilon(\kappa)$, we want to find a not necessarily extensional choice function f for a. We define $A(x, y, a)$ as the formula $\neg((x, y) \ \varepsilon \ a)$. By Proposition 4.6 we can define in $\mathcal{M}$ a functional relation f_A such that

$$\forall x \forall z \exists y (x, y) \ \varepsilon \ z \Rightarrow \exists \alpha \ \varepsilon \ \widehat{\kappa}(x, f_A(\alpha, y, a)) \ \varepsilon \ z$$

is realized. Since $\widehat{\kappa}$ is realized to be an ordinal, we can define in the pre-realizability model the required function f as follows:

$$(x, y) \ \varepsilon \ f \iff (x, y) \in a \wedge \exists \alpha \ \varepsilon \ \widehat{\kappa}(y = f_A(\alpha, x, a)) \wedge \forall \beta \ \varepsilon \ \widehat{\kappa}(\beta < \alpha \Rightarrow (x, f_A(\alpha, x, a)) \notin a)$$

Intuitively, this means that $f(x) = f_A(\alpha, x, a)$ for the least $\alpha < \widehat{\kappa}$ such that $f_A(\alpha, x, a)$ is defined. It should be clear that Func(f) is satisfied, hence f is as required. □

A similar argument shows that we even get non-extensional choice "for classes":

Corollary 4.8. *For each formula $A(\vec{w}, x)$, there exists a formula $A^*(\vec{w}, x)$ such that $\mathcal{N}_\varepsilon(\kappa)$ satisfies the following three formulas:*

- $\forall \vec{w}\ \forall x\ (A^*(\vec{w}, x) \Rightarrow A(\vec{w}, x))$,
- $\forall \vec{w}\ \forall x\ \forall y\ (A^*(\vec{w}, x) \Rightarrow A^*(\vec{w}, y) \Rightarrow x = y)$,
- $\forall \vec{w}\ (\exists x\ A(\vec{w}, x) \Rightarrow \exists x\ A^*(\vec{w}, x))$.

4.3. **Zorn's lemma in realizability models.**

Theorem 4.9. *For every $\alpha \leq \kappa$, $\mathcal{N}_\varepsilon(\kappa)$ satisfies a Zorn schema restricted to $\widehat{\alpha}$.*

Proof. Informally, the proof will proceed as follows: we are given binary relation R (which we will treat as if it were a strict ordering relation) such that every R-chain has a strict upper bound. For each $\alpha \in \widehat{\kappa}$, we must choose an $\simeq$-unique[8] y_α in such a way that y_α increases strictly with α.

The first issue we run into is that a priori, it is not possible to make such choices in a way which is compatible with $\simeq$. Fortunately, since $\widehat{\kappa}$ is a non-extensional ordinal, it ε-contains an $=$-unique representative of each of its $\in$-elements (Proposition 3.6). Therefore, it suffices to choose for each $\alpha\ \varepsilon\ \widehat{\kappa}$ an $=$-unique y_α. Then, for each $\alpha' \in \widehat{\kappa}$, it will suffices to let $y_{\alpha'} = y_\alpha$, where α is the unique ε-element of $\widehat{\kappa}$ such that $\alpha \simeq \alpha'$.

In order to choose y_α, we will need to proceed by induction over α: if y_β has been chosen for all $\beta\ \varepsilon\ \alpha$, then it suffices to choose y_α as one of the strict upper bounds of $\{y_\beta;\ \beta\ \varepsilon\ \alpha\}$, which is an R-chain. However, the next issue we run into is that ZF_ε does not a priori allow this kind of "construction by non-extensional induction": we will therefore have to do this inductive construction for "'outside the model", *i.e.* at the level of names in the ground model. In particular, a key step of the below proof is to be able to name, for each $\alpha \leq \kappa$, a unique representative of the R-chain $\{y_\beta;\ \beta\ \varepsilon\ \alpha\}$: this will be the role of the sets F_α, which will be defined below.

Recall that img is the class function from $\mathcal{M}$ to $\mathcal{M}$ defined by $\mathrm{img}(f) := \{(y, \pi);\ \exists x\ (\mathrm{pair}(x, y), \pi) \in f\}$ and that the formula $\forall f \forall y\ (y\ \varepsilon\ \mathrm{img}(f) \iff \exists x\ \mathrm{pair}(x, y)\ \varepsilon\ f)$ is true in $\mathcal{N}_\varepsilon(\kappa)$.

Let $R(\vec{w}, x, y)$ be any formula extensional with respect to x and y. For clarity, we will assume that the list of parameters $\vec{w}$ is empty: the proof in the general case is similar, though much less readable.

Let $\mathrm{UpperImg}(f, m)$ be the formula $\mathrm{Upper}^R(\mathrm{img}(f), m)$.

Let $\mathrm{UpperImg}^*(f, m)$ be the formula given by Corollary 4.8 (non-extensional choice for classes) applied to the formula $\mathrm{UpperImg}(f, m)$. Let μ be the function s_{UpperImg^*} from $\mathcal{M}$ to $\mathcal{M}$ given by Lemma 2.6 (naming of singletons). The following formulas are true in $\mathcal{N}_\varepsilon(\kappa)$ (by definition of $\mathrm{Upper}^R(\mathrm{img}(f), m)$) :

- $\forall f\ (\exists m\ \mathrm{Upper}^R(\mathrm{img}(f), m) \Rightarrow \exists m\ (m\ \varepsilon\ \mu(f)))$,
- $\forall f\ \forall m\ (m\ \varepsilon\ \mu(f) \Rightarrow \mathrm{Upper}^R(\mathrm{img}(f), m))$,
- $\forall f\ \forall m\ \forall m'\ (m\ \varepsilon\ \mu(f) \Rightarrow m'\ \varepsilon\ \mu(f) \Rightarrow m = m')$.

Now, by induction, for every $\alpha \leq \kappa$ we define a set $F_\alpha \in \mathcal{M}$:

- $F_{\alpha+1} := F_\alpha \cup \{(\mathrm{pair}(\widehat{\alpha}, y), t \bullet \pi);\ (y, \pi) \in \mu(F_\alpha) \land t \Vdash \widehat{\alpha}\ \varepsilon\ \widehat{\kappa}\}$,
- for α limit, $F_\alpha := \bigcup_{\beta<\alpha} F_\beta$.

Then we let $Z_R(\alpha, y)$ denote the following formula:

$$\exists \alpha' \varepsilon \widehat{\kappa}\ \exists y'\ (\alpha' \simeq \alpha \land y' \simeq y \land \mathrm{pair}(\alpha', y')\ \varepsilon\ F_\kappa).$$

[8] In other words, if $\alpha_1 \in \widehat{\kappa}$ and $\alpha_2 \simeq \alpha_1$, we must have $y_{\alpha_2} \simeq y_{\alpha_1}$.

The following facts follow from the definition of F_α :

(a) for every $\alpha \leq \kappa$, the statement $\forall\delta\ \forall y\ (\delta \not\varepsilon\ \alpha \Rightarrow \text{pair}(\delta, y) \not\varepsilon\ F_\alpha)$ is realized (by $\lambda x\ .\lambda t.\ tx$)

(b) for every $\alpha < \kappa$, and every $y \in \mathcal{M}$, we have $\|(\widehat{\alpha}, y) \not\varepsilon\ F_\kappa\| = \|\widehat{\alpha}\ \varepsilon\ \widehat{\kappa} \Rightarrow y \not\varepsilon\ \mu(F_\alpha)\|$

(c) for $\delta < \alpha < \kappa$, and $z \in \mathcal{M}$, we have $\|(\widehat{\delta}, z) \not\varepsilon\ F_\kappa\| \equiv \|\,\text{pair}(\widehat{\delta}, z) \not\varepsilon\ F_\alpha\|$.

It follows that the following formulas are realized:

(d) $\forall\beta\ \forall y\ (\text{pair}(\beta, y)\ \varepsilon\ F_\kappa \Rightarrow \beta\ \varepsilon\ \widehat{\kappa})$

(e) $\forall\gamma\varepsilon\widehat{\kappa}\ \exists f \left(\begin{array}{l} \quad\ \forall\beta\ \forall y\ (\text{pair}(\beta, y)\ \varepsilon\ f \Rightarrow \beta\ \varepsilon\ \gamma) \\ \wedge\ \ \forall z\ (\text{pair}(\gamma, z)\ \varepsilon\ F_\kappa \iff z\ \varepsilon\ \mu(f)) \\ \wedge\ \ \forall\beta\varepsilon\gamma\ \forall y\ (\text{pair}(\beta, y)\ \varepsilon\ F_\kappa \iff \text{pair}(\beta, y)\ \varepsilon\ f) \end{array}\right)$

(In (e), for all γ, the corresponding f is simply F_γ.)

We can then prove the following claim.

Claim 4.10. The following formulas are true in $\mathcal{N}_\varepsilon(\kappa)$:

(1) $\forall c\ (\text{Chain}^R(c) \Rightarrow \exists m\, \text{Upper}^R(c, m)) \Rightarrow \forall\gamma \in \widehat{\kappa}\ (\exists y\ Z^R(\gamma, y))$

(2) $\forall\beta{\in}\widehat{\kappa}\ \forall\gamma{\in}\widehat{\kappa}\ \forall y\ \forall z\ (Z_R(\beta, y) \Rightarrow Z_R(\gamma, z) \Rightarrow \beta \in \gamma \Rightarrow R(y, z)))$

(3) $\forall\beta{\in}\widehat{\kappa}\ \forall\gamma{\in}\widehat{\kappa}\ \forall y\ \forall z\ (Z_R(\beta, y) \Rightarrow Z_R(\gamma, z) \Rightarrow \beta \simeq \gamma \Rightarrow y \simeq z)$

Let us reason from within $\mathcal{N}_\varepsilon(\kappa)$:

Proof of (3): let y, z be sets and let $\beta, \gamma \in \widehat{\kappa}$ be such that $\beta \simeq \gamma$ and such that $Z_R(\beta, y)$ and $Z_R(\gamma, z)$ are true. Let $\beta' \simeq \beta$, $\gamma' \simeq \gamma$, $y' \simeq y$ and $z' \simeq z$ be such that $\beta'\ \varepsilon\ \widehat{\kappa}$, $\gamma'\ \varepsilon\ \widehat{\kappa}$, $\text{pair}(\beta', y')\ \varepsilon\ F_\kappa$ and $\text{pair}(\gamma', z')\ \varepsilon\ F_\kappa$. Then $\beta' \simeq \beta \simeq \gamma \simeq \gamma'$, $\beta'\ \varepsilon\ \kappa$ and $\gamma'\ \varepsilon\ \kappa$, so $\beta' = \gamma'$. Let f be such that $\forall x\ ((\gamma', x)\ \varepsilon\ F_\kappa \iff x\ \varepsilon\ \mu(f))$ (the existence of such a function f is guaranteed by (e) above). Then $y'\ \varepsilon\ \mu(f)$ and $z'\ \varepsilon\ \mu(f)$, so $y' = z'$ and $y \simeq z$.

Proof of (2): Let y, z be sets and let $\beta, \gamma \in \widehat{\kappa}$ be such that $\beta \in \gamma$ and that $Z_R(\beta, y)$ and $Z_R(\gamma, z)$ are true. Let $\beta' \simeq \beta$, $\gamma' \simeq \gamma$, $y' \simeq y$ and $z' \simeq z$ be such that $\beta'\ \varepsilon\ \widehat{\kappa}$, $\gamma'\ \varepsilon\ \widehat{\kappa}$, $\text{pair}(\beta', y')\ \varepsilon\ F_\kappa$ and $\text{pair}(\gamma', z')\ \varepsilon\ F_\kappa$: then $\beta'\ \varepsilon\ \gamma'$. (Indeed, $\beta' \in \gamma'$, so there exists $\beta''\ \varepsilon\ \gamma'$ such that $\beta'' \simeq \beta'$. Since $\beta'\ \varepsilon\ \widehat{\kappa}$ and $\beta''\ \varepsilon\ \widehat{\kappa}$ – because $\widehat{\kappa}$ is a transitive set –, $\beta' = \beta''$, so $\beta'\ \varepsilon\ \gamma'$.) Let f be such that $\text{pair}(\gamma', z')\ \varepsilon\ F_\kappa \iff z'\ \varepsilon\ \mu(f)$ and $\text{pair}(\beta', y')\ \varepsilon\ F_\kappa \iff \text{pair}(\beta', y')\ \varepsilon\ f$. Then on the one hand $z'\ \varepsilon\ \mu(f)$, so $\text{Upper}^R(\text{img}(f), z')$ is true, and on the other hand $\text{pair}(\beta', y')\ \varepsilon\ f$, so $R(y', z')$ is true. $R(y, z)$ is extensional with respect to y and z, thus $R(y, z)$ is true.

Proof of (1): let us assume that $\forall c\ (\text{Chain}^R(c) \Rightarrow \exists m\ \text{Upper}^R(c, m))$ holds, and let $\gamma \in \widehat{\kappa}$. Let $\gamma'\ \varepsilon\ \widehat{\kappa}$ be such that $\gamma' \simeq \gamma$. Let f be such that the following formulas hold:

(i) $\forall\beta\ \forall y'\ (\text{pair}(\beta, y')\ \varepsilon\ f \Rightarrow \beta\ \varepsilon\ \gamma')$,

(ii) $\forall z'\ (\text{pair}(\gamma', z')\ \varepsilon\ F_\kappa \iff z'\ \varepsilon\ \mu(f))$,

(iii) $\forall\beta\varepsilon\gamma'\ \forall y'\ (\text{pair}(\beta, y')\ \varepsilon\ F_\kappa \iff \text{pair}(\beta, y')\ \varepsilon\ f)$.

It suffices to prove that $\text{Chain}^R(\text{img}(f))$ is true, because it means that $\exists m\ \text{Upper}^R(c, m)$ is also true, which means that $\mu(f)$ is non-empty, and thus (by (ii)) that there exists z' such that $\text{pair}(\gamma', z')\ \varepsilon\ F_\kappa$.

Let $y, z \in \text{img}(f)$ be such that $y \not\simeq z$: we need to prove that $R(y, z)$ or $R(z, y)$ holds. Let $y', z'\ \varepsilon\ \text{img}(F)$ be such that $y' \simeq y$ and $z' \simeq z$. Let β and δ be such that $\text{pair}(\beta, y')\ \varepsilon\ f$ and $\text{pair}(\delta, z')\ \varepsilon\ F$. By (i), $\beta\ \varepsilon\ \gamma'$ and $\delta\ \varepsilon\ \gamma'$. By (iii), $\text{pair}(\beta, y')\ \varepsilon\ F_\kappa$ and $\text{pair}(\delta, z')\ \varepsilon\ F_\kappa$, so $Z_R(\beta, y)$ and $Z_R(\delta, z)$ are true. Therefore, by (3), $\beta \not\simeq \delta$. Since γ' is an ordinal, β and δ are ordinals, so we get $\beta \in \delta$ or $\delta \in \beta$. In the first case $R(y, z)$ holds by (2), in the second case $R(z, y)$ holds.

This completes the proof of the theorem. □

Corollary 4.11. *For every $\alpha \leq \kappa$, $\mathcal{N}(\kappa)$ satisfies Zorn's lemma restricted to $\widehat{\alpha}$.*

5. Preserving cardinals

We showed how to build for every ordinal κ a realizability model where κ has a representative $\widehat{\kappa}$ such that Zorn's lemma restricted to $\widehat{\kappa}$ holds. However, even if κ is a cardinal, $\widehat{\kappa}$ is not necessarily one. For instance, if we take for κ the cardinal $\aleph_1$, then $\widehat{\kappa}$ may become a countable ordinal. In the case of forcing models, two crucial properties are often used to solve this problem: chain conditions and closure. Any forcing that has the μ-chain condition for some $\mu \leq \kappa$ or μ-closure for some regular $\mu \geq \kappa$ preserves κ as a cardinal in the forcing model (for more detail see *e.g.* Kunen [7]). In this section, we present a general technique for preserving cardinals in realizability models: we introduce a special instruction that we denote η which will ensure that if κ is a cardinal in the ground model, then $\widehat{\kappa}$ is also a cardinal in the realizability model.

From now on, we assume that κ is an infinite cardinal. Let η and φ be two special instructions distinct from each other and from χ. Let $(\gamma_\alpha)_{\alpha<\kappa}$ be a family of special instructions pairwise distinct and different from χ, η and φ.

Notation 5.1. If P, Q and $\perp\!\!\!\perp$ are three sets of processes, we write "$P \succ_{\perp\!\!\!\perp} Q$" for "$Q \subseteq \perp\!\!\!\perp$ implies $P \cap \perp\!\!\!\perp \neq \emptyset$".

We take $\perp\!\!\!\perp$, $\mathcal{A}_\kappa^{\perp\!\!\!\perp}$, $\mathcal{N}_\varepsilon(\kappa)$ and $\mathcal{N}(\kappa)$ as in sections 3 and 4. However, in addition to being a final segment, we now assume that $\perp\!\!\!\perp$ satisfies the following properties:

- for every subset $U \subseteq \kappa$ such that the cardinality of $(\kappa \setminus U)$ is strictly less than κ,
$$\{\ \varphi * t \bullet \pi\ \} \succ_{\perp\!\!\!\perp} \{\ t * \gamma_\alpha \bullet \pi;\ \alpha \in U\ \},$$
- for every infinite set of terms A,
$$\{\ \eta * t \bullet a \bullet \pi;\ a \in A\ \} \succ_{\perp\!\!\!\perp} \{\ t * a \bullet b \bullet \pi;\ a, b \in A,\ a \neq b\ \}.$$

The first condition is required for technical reasons. The second is to be thought of as an analogue of the $\aleph_0$-chain condition. Note that although the μ-chain condition makes a forcing notion uninteresting unless $\mu \geq \aleph_1$, it is not the case for realizability.

We are going to show that in that case, $\widehat{\kappa}$ is not collapsed, that is there is no (extensional) surjection from an $\in$-element of $\widehat{\kappa}$ to $\widehat{\kappa}$. Since $\widehat{\kappa}$ is an ordinal in the non-extensional sense, it is enough to show that there is no non-extensional surjection from an ε-element of $\widehat{\kappa}$ to $\widehat{\kappa}$.

For every formula $F(x, y, \vec{w})$ of the language of realizability, we write $\mathrm{FunRel}_F(\vec{w})$ for the formula that says that F is a non-extensional functional binary relation, and we denote by $\mathrm{Surj}_F(a, b, \vec{w})$ the formula that says that F is surjective from a to b in the sense of the non extensional membership relation. More precisely, $\mathrm{FunRel}_F(\vec{w})$ is the formula $\forall x\ \forall y\ \forall y'\ (F(x, y, \vec{w}) \Rightarrow F(x, y', \vec{w}) \Rightarrow y \neq y' \Rightarrow \perp)$, while $\mathrm{Surj}_F(a, b, \vec{w})$ is the formula $\forall y(\forall x(F(x, y, \vec{w}) \Rightarrow x \not\varepsilon\, a) \Rightarrow y \not\varepsilon\, b)$.

Theorem 5.2. *For every formula $F(x, y, \vec{w})$ in the language of realizability, the formula $\forall \vec{w} \forall a^{\widehat{\kappa}} (\mathrm{FunRel}_F(\vec{w}) \Rightarrow \mathrm{Surj}_F(a, \widehat{\kappa}, \vec{w}) \Rightarrow \perp)$ is realized.*

Proof. We define the following terms:

- $\theta_2 := k(\eta f z(\lambda r.r))$

- $\theta_1 := \lambda z.k(fzz\theta_2)$
- $\theta_0 := \lambda a.\lambda f.\lambda s.\varphi(\lambda b.\,\text{cc}(\lambda k.s\theta_1 b))$

We are going to show that $\theta_0 \Vdash \forall\vec{w}\,\forall\mu^{\widehat{\kappa}}(\text{FunRel}_F(\vec{w}) \Rightarrow \text{Surj}_F(\mu,\widehat{\kappa},\vec{w}) \Rightarrow \bot)$. In order to simplify the notation, we omit the parameters $\vec{w}$. We fix $\mu < \kappa$, and we let $t, u \in \Lambda$ and $\pi \in \Pi$ such that $t \Vdash \text{FunRel}_F$, $u \Vdash \text{Surj}_F(\widehat{\kappa},\widehat{\lambda})$. We suppose by contradiction that $\theta_0 * \nu_\mu \bullet t \bullet u \bullet \pi \notin \perp\!\!\!\perp$. We let $v := \lambda b.\,\text{cc}(\lambda k.u\theta_1[f := t, s := u]b)$. To simply the notation we write θ_1 for $\theta_1[f := t, k := k_\pi]$ and θ_2 for $\theta_2[f := t, k := k_\pi]$. We let $U := \{\ \alpha < \kappa;\ v * \gamma_\alpha \bullet \pi \in \perp\!\!\!\perp\ \}$. We have $\varphi * v \bullet \pi \notin \perp\!\!\!\perp$, thus by the condition imposed on $\perp\!\!\!\perp$ for φ, the cardinality of $(\kappa\backslash U)$ is equal to κ. We let $B := \{\ \beta;\ \exists\alpha \in \kappa\backslash U\ \nu_\beta = \gamma_\alpha\ \}$. Then B is a subset of κ with cardinality κ.

For every $\beta \in B$, we have $v * \nu_\beta \bullet \pi \notin \perp\!\!\!\perp$, hence $u * \theta_1 \bullet \nu_\beta \bullet \pi \notin \perp\!\!\!\perp$. We know that $u \Vdash \forall x(F(x,\widehat{\beta}) \Rightarrow x \notin \widehat{\mu}) \Rightarrow \widehat{\beta} \notin \widehat{\kappa}$ and $\nu_\beta \bullet \pi \in \|\widehat{\beta} \notin \widehat{\kappa}\|$, therefore $\theta_1 \not\Vdash \forall x(F(x,\widehat{\beta}) \Rightarrow x \notin \widehat{\mu})$. It follows that there exists $\alpha_\beta < \mu$, $\zeta_\beta \in |F(\widehat{\alpha_\beta},\widehat{\beta})|$ and $\pi'_\beta \in \Pi$ such that $\theta_1 * \zeta_\beta \bullet \nu_{\alpha_\beta} \bullet \pi'_\beta \notin \perp\!\!\!\perp$. If we let $\theta_{2,\beta} := \theta_2[z := \zeta_\beta]$, this implies that $t * \zeta_\beta \bullet \zeta_\beta \bullet \theta_{2,\beta} \bullet \pi \notin \perp\!\!\!\perp$.

We show that $Z := \{\zeta_\beta;\ \beta \in B\}$ has size κ in the ground model. Suppose by contradiction that Z has size less than κ. For every $\zeta \in Z$, we let $B_\zeta = \{\beta \in B :\ \zeta_\beta = \zeta\}$. Since B has size $\kappa > \mu$ and $B = \bigcup_{\zeta\in Z} B_\zeta$, the set Z must contain an element ζ such that B_ζ has size bigger than μ. For all $\beta, \beta' \in B_\zeta$ such that $\alpha_{\beta'} = \alpha_\beta$, we have $\zeta_\beta = \zeta = \zeta_{\beta'}$ so $\zeta_\beta \Vdash F(\widehat{\alpha_\beta},\widehat{\beta})$, $\zeta_\beta \Vdash F(\widehat{\alpha_\beta},\widehat{\beta'})$ and $t \Vdash F(\alpha_\beta,\beta) \Rightarrow F(\alpha_\beta,\beta') \Rightarrow \beta \neq \beta' \Rightarrow \bot$; moreover, we have $t * \zeta_\beta \bullet \zeta_\beta \bullet \theta_{2,\beta} \bullet \pi \notin \perp\!\!\!\perp$, thus $\theta_{2,\beta} \notin |\beta \neq \beta'|$, which implies that $\beta = \beta'$. Hence $\beta \mapsto \alpha_\beta$ defines in the ground model an injection from B_ζ to μ, contradiction.

Since Z has size κ, there is a set $B_0 \subseteq B$ of size κ such that for every $\beta, \beta' \in B_0$, if $\beta \neq \beta'$, then $\zeta_\beta \neq \zeta_{\beta'}$. For every $\beta \in B_0$, we have $t * \zeta_\beta \bullet \zeta_\beta \bullet \theta_{2,\beta} \bullet \pi \notin \perp\!\!\!\perp$, while $t \Vdash F(\widehat{\alpha_\beta},\widehat{\beta}) \Rightarrow F(\widehat{\alpha_\beta},\widehat{\beta}) \Rightarrow \widehat{\beta} \neq \widehat{\beta} \Rightarrow \perp\!\!\!\perp$ and $\zeta_\beta \Vdash F(\widehat{\alpha},\widehat{\beta})$. Therefore, $\theta_{2,\beta} \not\Vdash \widehat{\beta} \neq \widehat{\beta}$. It follows that there exists a stack π''_β such that $\theta_{2,\beta} * \pi''_\beta \notin \perp\!\!\!\perp$, hence $\eta * t \bullet \zeta_\beta \bullet (\lambda r.r) \bullet \pi \notin \perp\!\!\!\perp$.

For every infinite set $B_1 \subseteq B_0$, the set $\{\zeta_\beta :\ \beta \in B_1\}$ is also infinite, hence by the restriction we imposed on $\perp\!\!\!\perp$ for η, there exists $\beta, \beta' \in B_1$ such that $\zeta_\beta \neq \zeta_{\beta'}$ and $t * \eta_\beta \bullet \eta_{\beta'} \bullet (\lambda r.r) \bullet \pi \notin \perp\!\!\!\perp$. Since $\zeta_\beta \neq \zeta_{\beta'}$, $\beta \neq \beta'$, so $(\lambda r.r)$ realizes $\beta \neq \beta'$. Moreover, we know that $\eta_\beta \Vdash F(\widehat{\alpha_\beta},\widehat{\beta})$, $\eta_{\beta'} \Vdash F(\widehat{\alpha_{\beta'}},\widehat{\beta'})$ and $t * \eta_\beta \bullet \eta_{\beta'} \bullet (\lambda r.r) \bullet \pi \notin \perp\!\!\!\perp$. Now, we know that for all $\alpha < \kappa$, $t \Vdash F(\widehat{\alpha},\widehat{\beta}) \Rightarrow F(\widehat{\alpha},\widehat{\beta'}) \Rightarrow \widehat{\beta} \neq \widehat{\beta'} \Rightarrow \bot$, so if α_β and $\alpha_{\beta'}$ were equal, we would obtain a contradiction by letting $\alpha := \alpha_\beta = \alpha_{\beta'}$.

In other words, if we say that (α,β) and (α',β') are *incompatible* when $\alpha = \alpha'$ and $\beta \neq \beta'$, then the set $\{(\alpha_\beta,\beta);\ \beta \in B_0\}$ cannot have an infinite antichain (that is an infinite set such that any two distinct elements are incompatible).

It follows that for every $\alpha < \mu$, the set $Y_\alpha := \{\beta \in B_0;\ \alpha = \alpha_\beta\}$ is finite, contradicting the fact that $B_0 = \bigcup_{\alpha<\mu} Y_\alpha$ has cardinality $\kappa > \mu$. □

By combining this theorem with the main theorem of the previous section, we get the final result.

Corollary 5.3. *Let $\mathcal{M}$ be a model of* ZF *with a global choice function and let κ be an infinite cardinal in $\mathcal{M}$, then there is a realizability model where $\widehat{\kappa}$ is a cardinal and* $\text{ZL}_{\widehat{\kappa}}$ *holds.*

Proof. By Theorem 4.9, the algebra $\mathcal{A}$ determines a realizability model where $\text{ZL}_{\widehat{\kappa}}$ holds. Theorem 5.2 ensures that $\widehat{\kappa}$ is a cardinal in this model. □

Remark 5.4. As in section 3, letting $\perp\!\!\!\perp = \emptyset$ makes $\mathcal{A}_\kappa^{\perp\!\!\!\perp}$ consistent while satisfying all the required conditions. The authors conjecture that, in this case too, there exists a suitable $\perp\!\!\!\perp$ such that $\mathcal{A}_\kappa^{\perp\!\!\!\perp}$ has a pre-realizability model which satisfies "$\beth 2$ has more than 2 elements".

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